Around p-adic cohomologies

A relative theory of p-adic differential equations

Gilles Christol & Zoghman Mebkhout

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Thank you Bruno :

without your long birthday ...

this work would have progressed even slower ...

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Basic field

Let K be a p-adic field (i.e. a complete extension of $\mathbb{Q}_p$)

and let normalize the valuation on K by $|p| = \dfrac{1}{p}$.

We will say algebra for K-algebra.

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The condition 3 (multiplicativity of $\|.\|_1$) ensures that the completion of the field of quotients of B_1 is a p-adic field that will be denoted by E . Hence it makes available the powerfull theory of D.E. over E .

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Principle : quotient the basic example by an ideal $\mathfrak{a}$ of $K{<}\mathbf{y}{>}^{\dagger}$.

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Moreover both topologies on $B^{\dagger}$ are separated and do not depend on the "presentation" $\mathfrak{a} \to K\langle\mathbf{y}\rangle^{\dagger}$.

One dimensional examples

$$B_1 = \{ \text{ analytic elements in } \mathbb{C}\,(\text{finite union of residue classes}) \} ,$$

$$B_\tau = \{ \text{ analytic elements in } \mathbb{C}\,(\text{finite union of disks with radius } \frac{1}{\tau}) \} .$$

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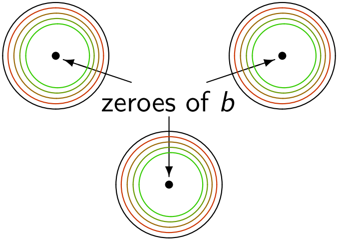
$B_\tau = \{$ analytic elements in $\complement$ (finite union of disks with radius $\dfrac{1}{\tau}$) $\}$.

They are special cases of "general examples" realized, for instance, with

$$m = 2\,,\ b \in K\!<\!y_1\!>,\ \mathfrak{a} = \left(1 - b\, y_2\right) K\!<\!y_1, y_2\!>$$



The B-algebra $\mathcal{A}_B(I)$ for a Banach algebra B

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It is a Frechet algebra for the two (equivalent) norm families :

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As usually these norms can be extended to the matrices :
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We will also denote by $\mathrm{Gl}\left(\mathcal{A}_B(I)\right)$ the group of invertible matrices.

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* The function $\exp(\pi\, y\, x) = \sum_{n\in\mathbb{N}} \frac{\pi^n}{n!}\, y^n\, x^n \in \mathcal{A}_{B^{>1}}(I)$,

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Main results in the null monodromy case

Let $B^\dagger$ be a $\dagger$-adic algebra and E the completion of the quotient field of B_1 .

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Theorem EB (from E to B_1)

Let $H \in \mathrm{Gl}\left(\mathcal{A}_{E}(I)\right)$ *be s.t.* $G \stackrel{def}{=} \frac{d}{dx} H\, H^{-1} \in \mathrm{Mat}\left(\mathcal{A}_{B_1}(I)\right)$,
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Putting the two theorems together gives

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Main results for general Robba differential modules

Theorem (structure of Robba modules over p-adic fields)

Let $G \in \mathrm{Mat}\left(\mathcal{A}_E(I)\right)$ *satisfying Robba condition* $[\ (\forall \rho \in I) \ \ \mathrm{ray}_G(\rho) = \rho\ ]$
with exponent $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_p^d/\mathfrak{E}$ *satisfying DNL,*
$[\widetilde{\alpha} = \{\alpha_i\} \in \left\{\mathbb{Z}_p/\mathbb{Z}\right\}^d$ and the Differences $\alpha_i - \alpha_j$ are Non Liouville numbers$]$
Then there is a "change-of-basis matrix" $H \in \mathrm{Gl}\left(\mathcal{A}_E(I)\right)$ *(from* G *to* $\frac{1}{x}M$*)*
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$M = D + N$ *with* $D = \mathrm{diag}\{\alpha_1, ..., \alpha_d\}$ *,* N *is nilpotent and* $DN = ND$*.*

Main results for general Robba differential modules

Theorem (structure of Robba modules over p-adic fields)

Let $G \in \mathrm{Mat}\left(\mathcal{A}_E(I)\right)$ *satisfying Robba condition* $[\ (\forall \rho \in I) \quad \mathrm{ray}_G(\rho) = \rho\ ]$
with exponent $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_p^d / \mathfrak{E}$ *satisfying DNL,*
$[\widetilde{\alpha} = \{\alpha_i\} \in \{\mathbb{Z}_p/\mathbb{Z}\}^d$ and the Differences $\alpha_i - \alpha_j$ are Non Liouville numbers$]$
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Theorem (structure of Robba modules over †-adic algebras)

Let $G \in \mathrm{Mat}\left(\mathcal{A}_{B^{>1}}(I)\right)$ *satisfying Robba condition with DNL exponent.*
Then the change-of-basis matrix H *can be taken in* $\mathrm{Gl}\left(\mathcal{A}_{B^{>1}}(I)\right)$ *.*

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Theorem WV : *Let $J' \subsetneq J$ two closed subintervals of I .*
If $H = \sum_{s\in\mathbb{Z}} H_s\, x^s \in \mathrm{Gl}\left(\mathcal{A}_{B_1}(J)\right)$ satisfies both conditions

WV 1. $G \overset{def}{=} \frac{d}{dx} H\, H^{-1} \in \mathrm{Mat}\left(\mathcal{A}_{B_\tau}(J)\right)$ *for some* $\tau > 1$,

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which itself is a consequence of the density of B_τ in B_1 .

Proof of WV : matrices $G_{<n>}$ and function $\mathrm{ray}_G(t,\rho)$

By WV1 , $H \in \mathrm{Gl}\left(\mathcal{A}_{B_1}(J)\right)$ and $G \stackrel{\mathrm{def}}{=} \frac{d}{dx} H\, H^{-1} \in \mathrm{Mat}\left(\mathcal{A}_{B_\tau}(J)\right)$.

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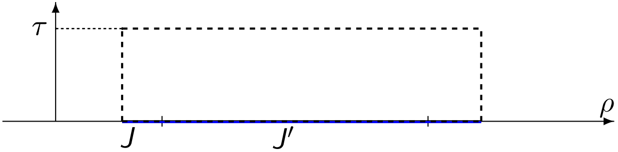

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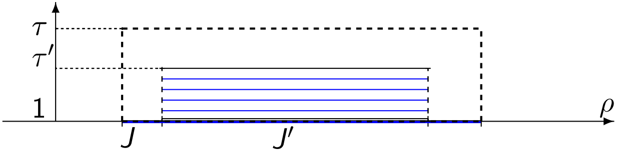
The theorem WV asserts that $\mathrm{ray}_G(t,\rho) = \rho$ on $[1,\tau'] \times J'$.



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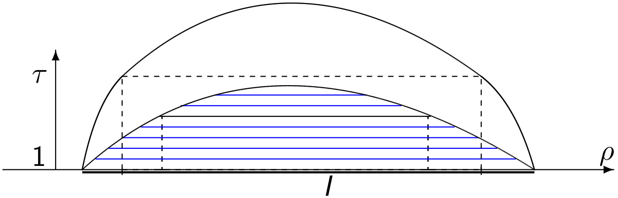

and the theorem BB asserts this is true on some open set edged with I .

Proof of WV : Frobenius machinery

By WV2 , $H = \sum_{s\in\mathbb{Z}} H_s\, x^s$ with $H_0 = \mathrm{Id}$, $\| H - \mathrm{Id} \|_{1,J} < 1$.

Then, $H^{(\ell)} \stackrel{\text{def}}{=} \sum_{s\in\mathbb{Z}} H_{p^\ell s}\, x^{p^\ell s} \in \mathrm{Gl}\left(\mathcal{A}_{B_1}(J)\right)$ for $\ell \geq 1$.

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Unfortunately $\lim_{\ell\to\infty} \tau_\ell = 1$ and we have ... yet to work.

Proof of WV : going from $R^{(\ell)}$ to $R^{(\ell+1)}$

Let's define $G^{(\ell)}$ by $\quad p^\ell x^{p^\ell} G^{(\ell)}(x^{p^\ell}) \stackrel{\text{def}}{=} x \frac{d}{dx} H^{(\ell)} {H^{(\ell)}}^{-1}$.

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Then $G^{(\ell)} \in \mathrm{Mat}\left(\mathcal{A}_{B_1}(J^{p^\ell})\right)$ and $\quad R_G^{(\ell+1)} = R^{(1)}_{G^{(\ell)}}(x^{p^\ell})\; R_G^{(\ell)}$.

Lemma RG: If $\tau_\ell \leq \tau$, $R_G^{(\ell)} \in \mathrm{Gl}\left(\mathcal{A}_{B_{\tau_\ell}}(J)\right) \Rightarrow G^{(\ell)} \in \mathrm{Mat}\left(\mathcal{A}_{B_{\tau_\ell}}(J^{p^\ell})\right)$.

Proof : $\quad p^\ell x^{p^\ell} G^{(\ell)}(x^{p^\ell}) = x\left(\frac{d}{dx} R_G^{(\ell)} + R_G^{(\ell)}\; G\right) {R_G^{(\ell)}}^{-1}$.

Lemma GR : *If* $R_G^{(\ell)} \in \mathrm{Gl}\left(\mathcal{A}_{B_{\tau_\ell}}(J)\right)$ *and* $\|x\, G^{(\ell)}\|_{\tau_\ell, J^{p^\ell}} < 1$

then $R_G^{(\ell+1)} \in \mathrm{Gl}\left(\mathcal{A}_{B_{\tau_\ell}}(J)\right)$ *and* $\|x\, G^{(\ell+1)}\|_{\tau_\ell, J^{p^{\ell+1}}} < p$.

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Proof: Noticing that $\left| \sum_{\zeta^p=1} \frac{(\zeta-1)^n}{n!} \right| < 1$ for $n \geq 1$, we get :

$$R^{(1)}_{G^{(\ell)}} = \sum_{n\in\mathbb{N}} \sum_{\zeta^p=1} \frac{(\zeta-1)^n}{n!}\, x^n\, G^{(\ell)}_{<n>} \in \mathrm{Gl}\left(\mathcal{A}_{B_{\tau_\ell}}(J^{p^\ell})\right) ,$$

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It remains to kill the "p" that appears at each step ℓ .

Proof of WV : killing p in nonresidue terms of $G^{(\ell)}$

Principle : reducing the interval J .

Let's write $x^{p^\ell} G^{(\ell)}(x^{p^\ell}) = \sum G_s^{(\ell)} x^{sp^\ell}$, $J = [p^a, p^b]$, $J' = [p^{a'}, p^{b'}]$ and let's choose ℓ_0 big enough to have $\sum_{\ell \geq \ell_0} p^{-\ell} \leq \min\{b - b' ; \ a' - a\}$.

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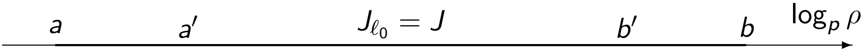

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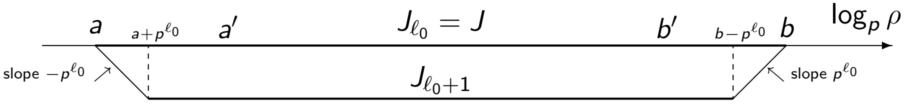

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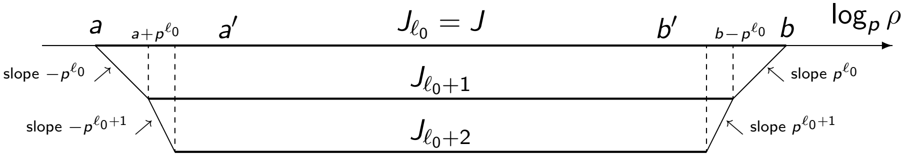

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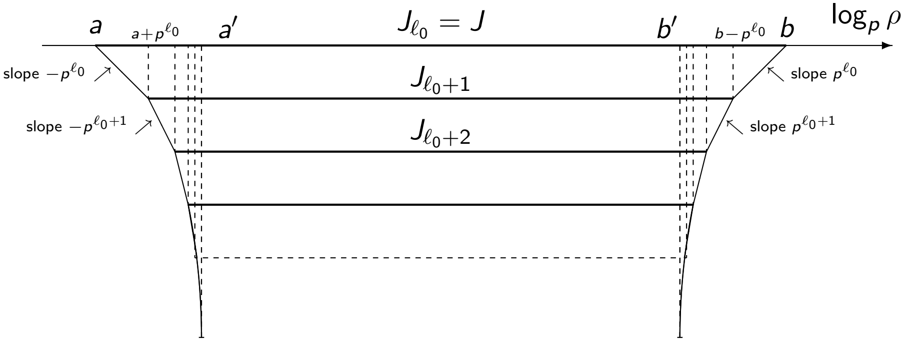

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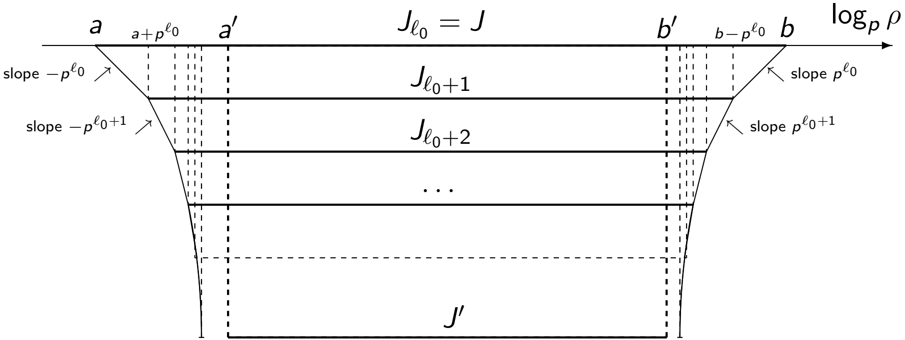

Proof of WV : killing p in the residue of $G^{(\ell)}$

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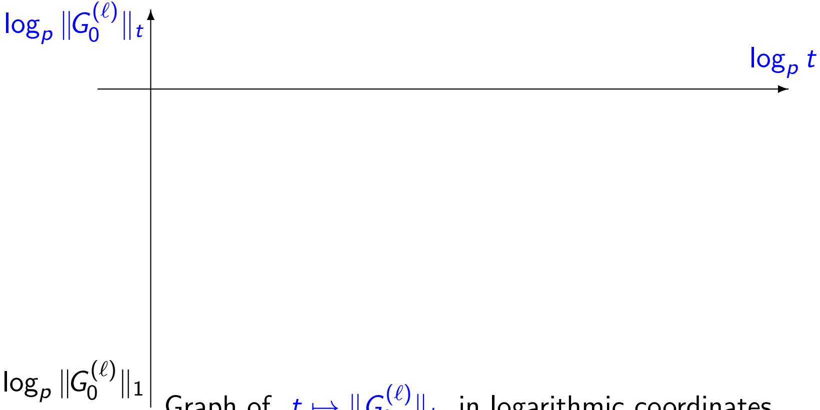

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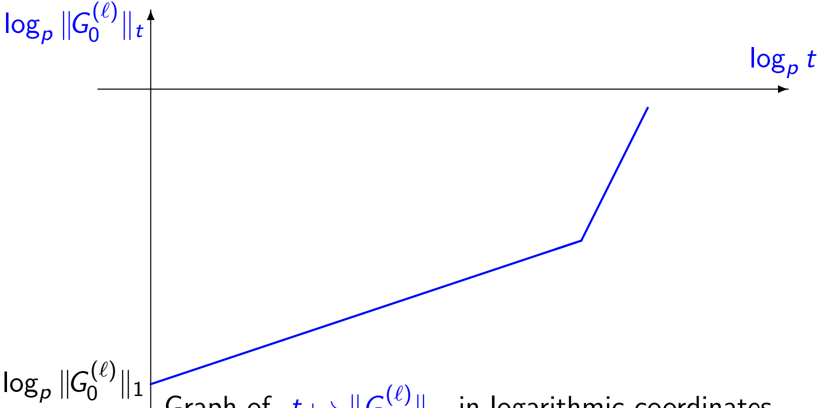

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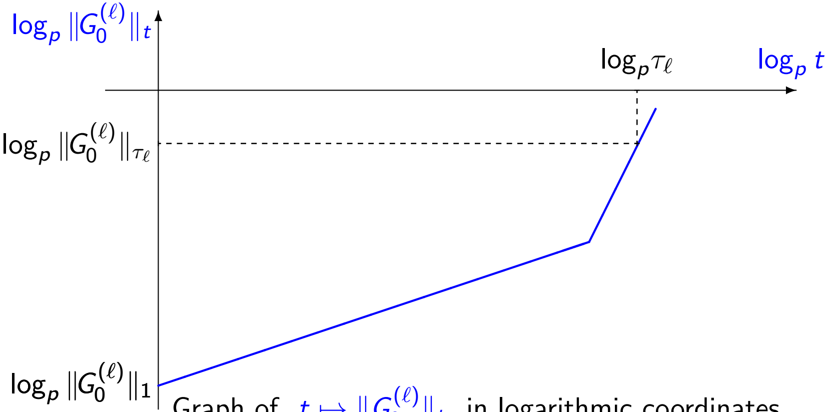

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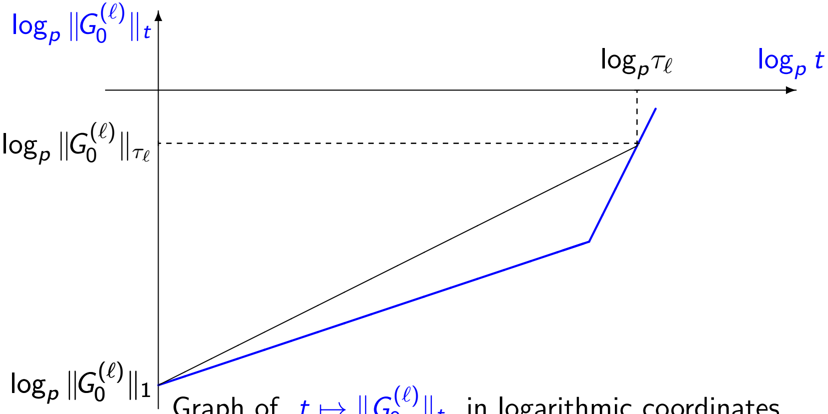

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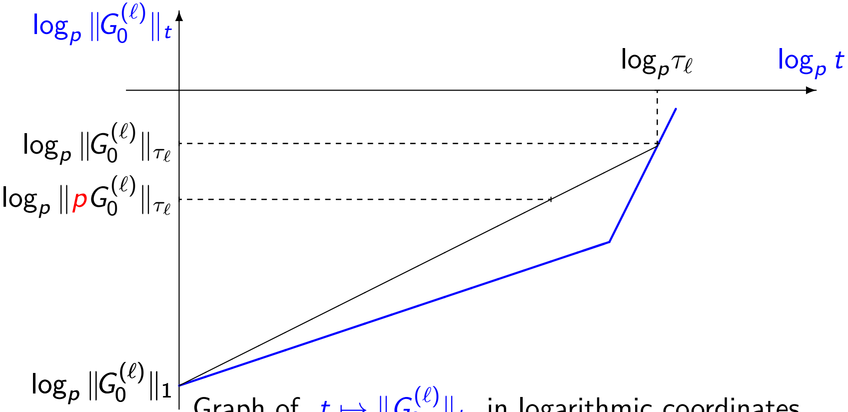

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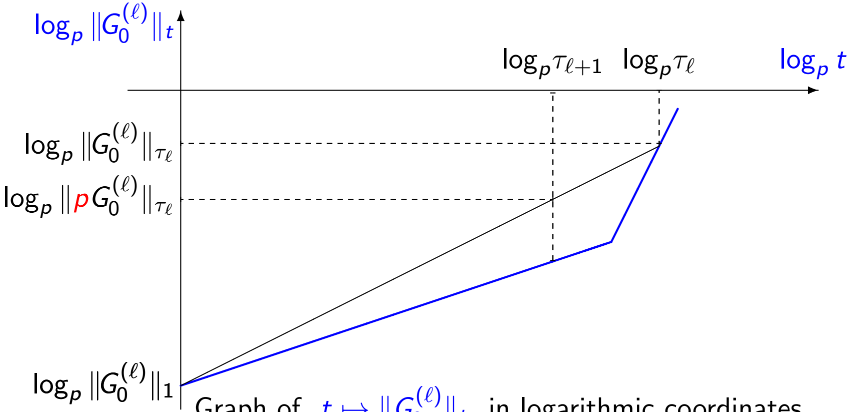

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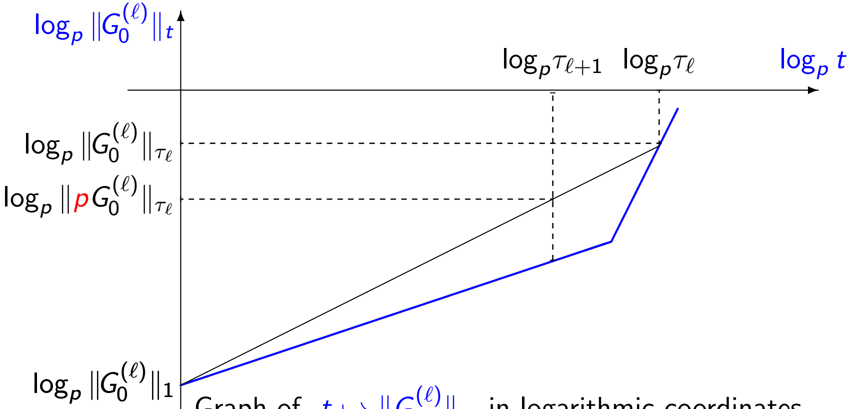

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We conclude showing $\tau' \stackrel{\text{def}}{=} \lim_{\ell\to\infty} \tau_\ell > 1$.

Thank you for your attention