Generic Singularities of the 3D-Contact sub-Riemannian Conjugate Locus

B. Bonnet [a], J.P. Gauthier [a], F. Rossi [b]
[a] *Aix Marseille Université, CNRS, ENSAM, Université de Toulon, LIS, Marseille, France*
[b] *Dipartimento di Matematica "Tullio Levi-Civita" Università degli Studi di Padova, Padova, Italy*

Received *****; accepted after revision +++++
Presented by

Abstract

In this paper, we extend and complete the classification of the generic and stable singularities of the 3D-contact sub-Riemmanian conjugate locus in a neighbourhood of the origin.
To cite this article: B. Bonnet, J.P. Gauthier, F. Rossi, C. R. Acad. Sci. Paris, Ser. I ??? (201?).

Résumé

Singularités Génériques du Lieu Conjugé en Géométrie sous-Riemannienne dans le cas 3D-contact. Dans cet article, nous étendons et achevons la classification des singularités génériques du lieu conjugué sous-Riemannien 3D-contact au voisinage de l'origine. *Pour citer cet article : B. Bonnet, J.P. Gauthier, F. Rossi, C. R. Acad. Sci. Paris, Ser. I ? ? ? (201 ?).*

Version française abrégée

L'une des différences fondamentales entre les géométries Riemanniennes et sous-Riemanniennes réside dans le fait que les géodésiques sous-Riemanniennes ne sont génériquement pas localement optimales dans un voisinage de leur origine. Ce comportement étrange se traduit par la présence d'un grand nombre de singularités le long des surfaces générées par les points critiques de l'application exponentielle, que l'on appelle les *caustiques*. La première étude des singularités génériques les moins dégénérés de la caustique sous-Riemannienne 3D-contact a été présentée dans [1]. Elle fut suivie d'une analyse des premiers cas dégénérés génériques des *semi-caustiques*, qui sont les intersection de la caustique avec des demi-espaces bien choisis, menée indépendamment dans [2,6]. Cette étude fut complétée dans [3] par une classification

Email addresses: `benoit.bonnet@lis-lab.fr` (B. Bonnet), `jean-paul.gauthier@univ-tln.fr` (J.P. Gauthier), `francesco.rossi@math.unipd.it` (F. Rossi).

exhaustive des singularités génériques de ces semi-caustiques. Dans cet article, nous achevons cette classification en l'étendant à la caustique toute entière. Il est à noter que la *stabilité* de ces singularités est une question très délicate, qui n'a été abordée qu'ultérieurement dans [4] pour les cas les moins dégénérés.

Il a été démontré dans [3] que dans les situations dégénérées génériques, les intersections de la semi-caustique avec des plans horizontaux forment des courbes fermées présentant des *auto-intersections* (voir Figure 2-3 ci-dessous). Ces dernières peuvent être classifiées à l'aide de *symboles*. Ici, un symbole est un sextuplet de nombres $(s_1, \ldots, s_6)$ où chaque s_i est égal à la moitié du nombre d'auto-intersections le long de la portion de courbe entre deux points cuspidaux successifs. Cette notation nous permet de formuler le résultat principal de cet article.

Theorem 0.1 *Soit M une variété différentielle lisse et connexe de dimension 3 et $SubR(M)$ l'espace des distributions sous-Riemanniennes de contact sur M (voir Définition 2.1 ci-dessous) équipé de la topologie de Whitney. Il existe un ouvert dense $\mathscr{E} \subset SubR(M)$ tel que, pour toute distribution $(\Delta, \mathbf{g}) \in \mathscr{E}$, l'on a les faits suivants.*

(i) Il existe une courbe lisse $\mathscr{C} \subset M$ en dehors de laquelle les intersections de la caustique avec des plan horizontaux $\{h = \pm\epsilon\}$ sont des courbes fermées présentant quatre point cuspidaux (voir Figure 1-gauche ci-dessous).

(ii) Il existe un ouvert dense $\mathscr{O} \subset \mathscr{C}$ le long duquel les intersections de la caustique avec des plans horizontaux $\{h = \pm\epsilon\}$ sont décrites par des paires de symboles de la forme $(\mathscr{S}_i, \mathscr{S}_j)$ avec $i, j \in \{1, 2, 3\}$ (voir Figures 2,3-gauche), où

$$\mathscr{S}_1 = (0, 1, 1, 1, 1, 1), \ \mathscr{S}_2 = (2, 1, 1, 1, 1, 1), \ \mathscr{S}_3 = (2, 1, 1, 2, 1, 0).$$

(iii) Il existe un sous-ensemble discret $\mathscr{D} \subset \mathscr{C}$ complémentaire de $\mathscr{O}$ dans $\mathscr{C}$ le long duquel les intersections de la caustique avec des plans horizontaux $\{h = \pm\epsilon\}$ sont décrites par des paires de symboles de la forme $(\mathscr{S}_i, \mathscr{S}_j)$ avec $i \in \{1, 2, 3\}, j \in \{4, 5, 6, 7\}$ (voir Figures 2,3-centre & droite ci-dessous), où

$$\mathscr{S}_4 = (\tfrac{1}{2}, \tfrac{1}{2}, 1, 0, 0, 1), \ \mathscr{S}_5 = (1, \tfrac{1}{2}, \tfrac{1}{2}, 1, 1, 1), \ \mathscr{S}_6 = (\tfrac{3}{2}, \tfrac{1}{2}, 1, 1, 0, 1), \ \mathscr{S}_7 = (2, \tfrac{1}{2}, \tfrac{1}{2}, 2, 0, 0).$$

La preuve de ce résultat respose sur le concept de *coordonnées normales* en géométrie sous-Riemannienne, qui sont le pendant de celles classiquement définies en géométrie Riemanienne. Ces coordonnées permettent de définir une notion de *forme normale* pour les distributions sous-Riemaniennes de contact (voir Théorème 2.6 ci-dessous) dérivée dans [3,6]. Ce résultat structurel permet d'écrire toute métrique sous-Riemannienne 3D-contact comme une perturbation lisse de la métrique de Heisenberg. En combinant l'expression issue de cette forme normale à l'introduction de coordonnées renormalisées *adaptées*, il est possible de réecrire les *semi lieux conjugués* (voir Définition 2.5 ci-dessous) d'une distribution $(\Delta, \mathbf{g})$ sous la forme d'une *suspension*.

La classification des singularités génériques de la semi-caustique (voir Théorème 3.2 ci-dessous) a été obtenue dans [3] en étudiant l'ensemble des *auto-intersections* du semi lieu conjugué à l'aide de cette suspension et de théorèmes de *transversalité* (voir e.g. [7]). Nous achevons cette classification en démontrant que les semi lieux conjugués peuvent être générés de manière indépendante à partir des invariants apparaissant dans la forme normale.

1. Introduction

One of the major discrepancies between Riemannian and sub-Riemannian geometry is the fact that sub-Riemannian geodesics generically fail to be locally optimal in a neighbourhood of their initial point. This uncanny property translates into the presence of a wide amount of singularities along the so-called *caustic*

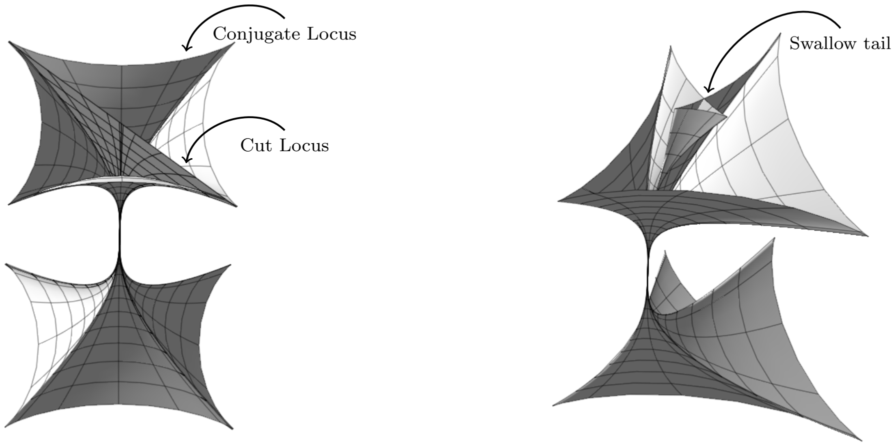

FIGURE 1. Generic conjugate and cut loci outside $\mathscr{C}$ (left) and appearance of the swallow tail near $\mathscr{C}$ (right).

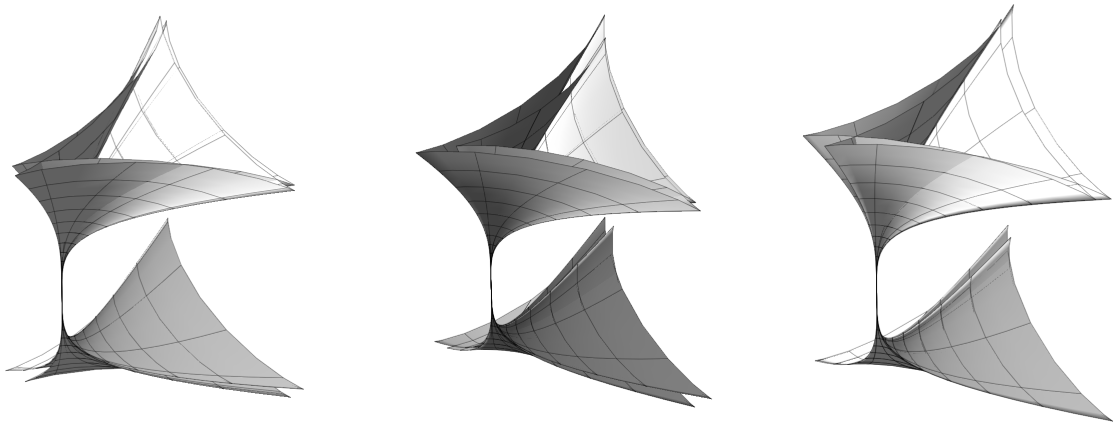

FIGURE 2. Drawing of generic conjugate loci on the curve $\mathscr{C}$ corresponding to the pairs $(\mathscr{S}_1, \mathscr{S}_2)$, $(\mathscr{S}_4, \mathscr{S}_1)$ and $(\mathscr{S}_2, \mathscr{S}_6)$

surfaces generated by the critical points of the corresponding exponential map. The analysis of the least degenerate generic behaviour of the 3D-contact sub-Riemannian caustic was carried out for the first time in the seminal paper [1]. The further degenerate generic situations were then studied independently in [2,6], and the full classification of the generic singularities of the *semi-caustics* - i.e. the intersections of the caustic with adequate half-spaces - was completed in [3]. In the present note, we accomplish this research by providing a complete classification of the generic singularities of the full caustic. Let it be noted that the question of *stability* for these singularities is very delicate, and was only studied subsequently in [4].

It was shown in [3] that for the generic degenerate situations, the intersection of the caustic with a horizontal plane is a closed curve with six cusp points (see Figures 2 above and 3 below). Such curves can be characterized by their *self-intersections*, which can be described by *symbols*. Here, a symbol is a six-tuple of numbers $(s_1, ..., s_6)$ where each s_i is half the number of self-intersections appearing along the piece of curve joining two consecutive cusp points. Using this notation, we can state the main result of this article.

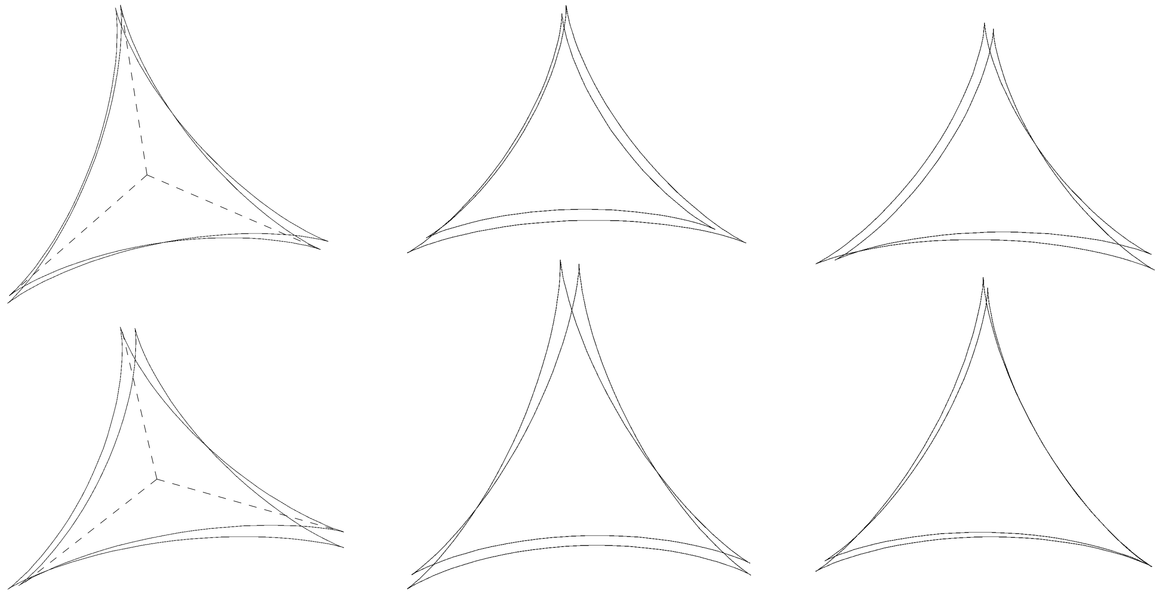

FIGURE 3. Intersections of the caustics corresponding to the symbols $(\mathscr{S}_1, \mathscr{S}_2)$, $(\mathscr{S}_4, \mathscr{S}_1)$ and $(\mathscr{S}_2, \mathscr{S}_6)$ with the planes $h = \pm\epsilon$ (upper and lower drawings respectively), and intersection of the corresponding cut loci (dashed lines left)

Theorem 1.1 (Main result) *Let M be a 3-dimensional smooth and connected manifold and $SubR(M)$ be the space of contact sub-Riemannian distributions over M endowed with the Withney topology. There exists an open and dense subset $\mathscr{E} \subset SubR(M)$ such that for any $(\Delta, \mathbf{g}) \in \mathscr{E}$ the following holds.*

(i) There exists a smooth curve $\mathscr{C} \subset M$ such that, outside $\mathscr{C}$, the intersections of the caustic with horizontal planes $\{h = \pm\epsilon\}$ are closed curves exhibiting 4 cusp points (see Figure 1-left above).

(ii) There exists an open and dense subset $\mathscr{O} \subset \mathscr{C}$ on which the intersections of the caustic with horizontal planes $\{h = \pm\epsilon\}$ are described by pairs of symbols $(\mathscr{S}_i, \mathscr{S}_j)$ with $i, j \in \{1, 2, 3\}$ (see Figures 2,3-left above) and

$$\mathscr{S}_1 = (0, 1, 1, 1, 1, 1),\ \mathscr{S}_2 = (2, 1, 1, 1, 1, 1),\ \mathscr{S}_3 = (2, 1, 1, 2, 1, 0).$$

(iii) There exists a discrete subset $\mathscr{D} \subset \mathscr{C}$ complement of $\mathscr{O}$ in $\mathscr{C}$ on which the intersections of the caustic with horizontal planes $\{h = \pm\epsilon\}$ are described by pairs of symbols $(\mathscr{S}_i, \mathscr{S}_j)$ with $i \in \{1, 2, 3\}, j \in \{4, 5, 6, 7\}$ (see Figures 2,3-center & right above) and

$$\mathscr{S}_4 = (\tfrac{1}{2}, \tfrac{1}{2}, 1, 0, 0, 1),\ \mathscr{S}_5 = (1, \tfrac{1}{2}, \tfrac{1}{2}, 1, 1, 1),\ \mathscr{S}_6 = (\tfrac{3}{2}, \tfrac{1}{2}, 1, 1, 0, 1),\ \mathscr{S}_7 = (2, \tfrac{1}{2}, \tfrac{1}{2}, 2, 0, 0).$$

This result is in a sense the most natural one to be expected after the classification of the half conjugate loci displayed in [3]. Indeed, it transcribes the fact that the upper and lower semi-caustics are independent and that there are no extra couplings appearing between the two structures. Indeed, the only possible obstruction to the combination of two given symbols is that the corresponding codimension in the space of Taylor coefficients - which is preserved by standard arguments of transversality theory - is strictly larger than 3. In particular, this generically prevents pairs of the form $(\mathscr{S}_i, \mathscr{S}_j)$ with $i, j \in \{4, 5, 6, 7\}$ from appearing.

2. 3D-Contact sub-Riemannian manifolds and their conjugate locus

In this section, we recall some elementary facts about sub-Riemannian geometry defined over 3-dimensional manifolds. For a complete introduction, see [5].

Definition 2.1 (Sub-Riemannian manifold) *A 3D-contact sub-Riemannian manifold is defined by a triple* $(M, \Delta, \mathbf{g})$ *where*

- M *is an* 3*-dimensional smooth and connected differentiable manifold,*
- Δ *is a smooth* 2*-dimensional distribution over* M *with step 1, i.e.*

$$\mathrm{Span}\,\{X_1(q), X_2(q), [X_1(q), X_2(q)]\} = T_qM,$$

  for all $q \in M$ *and* $(X_1(q), X_2(q))$ *spanning* $\Delta(q)$*.*
- $\mathbf{g}$ *is a Riemannian metric over* M*.*

Definition 2.2 (Horizontal curves and sub-Riemannian metric) *An absolutely continuous curve* $\gamma(\cdot)$ *is said to be* horizontal *if* $\dot\gamma(t) \in \Delta(\gamma(t))$ *for* $\mathscr{L}^1$*-almost every* $t \in [0, T]$*. We define the* length $l(\gamma(\cdot))$ *of a horizontal curve* $\gamma(\cdot)$ *as*

$$l(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\,\mathrm{d}t.$$

For $(q_0, q_1) \in M$*, it is then possible to define the* sub-Riemannian distance $d_{\mathrm{sR}}(q_0, q_1)$ *as the infimum of the length of the horizontal curves connecting* q_0 *and* q_1*.*

Given a local orthonormal frames (X_1, X_2) for the metric $\mathbf{g}$ which spans Δ, the Carnot-Carathéodory distance $d_{\mathrm{sR}}(q_0, q_1)$ can be alternatively computed by solving the optimal control problem

$$\min_{(u_1,u_2)} \int_0^T (u_1^2(t) + u_2^2(t))\mathrm{d}t \qquad \text{s.t.} \qquad \dot\gamma(t) = u_1 X_1(\gamma(t)) + u_2(t) X_2(\gamma(t)),\;\; u_1^2(t) + u_2^2(t) \le 1, \tag{1}$$

with the boundary constraints $(\gamma(0), \gamma(T)) = (q_0, q_1)$. We detail in the following Proposition the explicit form of 3D-contact sub-Riemannian geodesics obtained by applying the maximum principle to (1).

Proposition 2.3 (The Pontryagin Maximum Principle in the 3D contact case) *Let* $\gamma(\cdot) \in \mathrm{Lip}([0, T], M)$ *be a horizontal curve and* $\mathcal{H} : T^*M \to \mathbb{R}$ *be the Hamiltonian associated to the contact geodesic problem, defined by*

$$\mathcal{H}(q, \lambda) = \tfrac{1}{2}\langle \lambda, X_1(q) + X_2(q)\rangle,$$

for any $(q, \lambda) \in T^*M$*. Then, the curve* $\gamma(\cdot)$ *is a contact geodesic parametrized by sub-Riemannian arclength if and only if there exists a Lipschitzian curve* $t \in [0, T] \mapsto \lambda(t) \in T^*_{\gamma(t)}M$ *such that* $t \mapsto (\gamma(t), \lambda(t))$ *is a solution of the Hamiltonian system*

$$\dot\gamma(t) = \partial_\lambda \mathcal{H}(\gamma(t), \lambda(t)), \qquad \dot\lambda(t) = -\partial_q \mathcal{H}(\gamma(t), \lambda(t)), \qquad \mathcal{H}(\gamma(t), \lambda(t)) = \tfrac{1}{2}. \tag{2}$$

We denote by $\overrightarrow{\mathcal{H}} \in \mathrm{Vec}(T^*M)$ *the corresponding Hamiltonian vector field defined over the cotangent bundle and by* $(\gamma(t), \lambda(t)) = e^{t\overrightarrow{\mathcal{H}}}(q_0, \lambda_0)$ *the corresponding solution of* (2)*.*

Both the absence of abnormal lifts and the sufficiency of the maximum principle are consequences of the contact hypothesis made on the sub-Riemannian structure (see e.g. [5, Chapter 4]).

Definition 2.4 (Exponential map) *Let* $q_0 \in M$ *and* $\Lambda_{q_0} = \{\lambda \in T^*_{q_0}M \text{ s.t. } \mathcal{H}(q_0, \lambda_0) = \frac{1}{2}\}$*. We define the* exponential map *from* q_0 *as*

$$\mathrm{E}_{q_0} : (t, \lambda_0) \in \mathbb{R}_+ \times \Lambda_{q_0} \mapsto \pi_M\left(e^{t\overrightarrow{\mathcal{H}}}(q_0, \lambda_0)\right),$$

where $\pi_M : T^*M \to M$ *is the canonical projection.*

In this paper, we study the germ at the origin - i.e. equivalence classes of maps defined by equality of derivatives up to a certain order - of the *conjugate* and *cut loci* associated to contact sub-Riemannian structures.

Definition 2.5 (Cut and conjugate locus) *Let $(M,\Delta,\mathbf{g})$ be a 3D contact sub-Riemannian manifold, $(q_0,\lambda_0)\in T^*M$ and $\gamma(\cdot)=\mathrm{E}_{q_0}(\cdot,\lambda_0)$ be a geodesic parametrized by sub-Riemannian arclength. The* cut time *associated to $\gamma(\cdot)$ is defined by*

$$\tau_{\text{cut}}=\sup\{t\in\mathbb{R}_+ \ s.t.\ \gamma_{[0,t)}(\cdot) \text{ is optimal }\},$$

and the corresponding cut locus *is*

$$\mathrm{Cut}(q_0)=\{\gamma(\tau_{\text{cut}})\ s.t.\ \gamma(\cdot) \text{ is a sub-Riemannian geodesic from } q_0\}.$$

The first conjugate time τ_{conj} *associated to the curve $\gamma(\cdot)$ is define by*

$$\tau_{\text{conj}}=\inf\{t\in\mathbb{R}_+\ s.t.\ (t,\lambda_0) \text{ is a critical point of } \mathrm{E}_{q_0}(\cdot,\cdot)\},$$

and the corresponding conjugate locus *is then defined by*

$$\mathrm{Conj}(q_0)=\{\gamma(\tau_{\text{conj}})\ s.t.\ \gamma(\cdot) \text{ is a sub-Riemannian geodesic from } q_0\}.$$

We recall in Theorem 2.6 below the *normal form* introduced formally in [6] and then derived geometrically in [3] for 3D-contact sub-Riemannian structures. Up to a simple change of coordinates, we can assume that $0\in M$ and study the germ of the conjugate locus in a neighbourhood of the origin.

Theorem 2.6 (Normal form for 3D-contact sub-Riemannian distributions) *Let $(M,\Delta,\mathbf{g})$ be a 3D-contact sub-Riemannian structure and (X_1,X_2) be a local orthonormal frame for Δ in a neighbourhood of the origin. Then, there exists a smooth system of so-called* normal coordinates *(x,y,w) on M along with two maps $\beta,\gamma\in C^\infty(M,\mathbb{R})$ such that (X_1,X_2) can be written in* normal form *as*

$$\begin{cases} X_1(x,y,w)=(1+y^2\beta(x,y,w))\partial_x-xy\beta(x,y,w)\partial_y+\dfrac{y}{2}(1+\gamma(x,y,w))\partial_w,\\ X_2(x,y,w)=(1+x^2\beta(x,y,w))\partial_y-xy\beta(x,y,w)\partial_x-\dfrac{x}{2}(1+\gamma(x,y,w))\partial_w,\\ \beta(0,0,w)=\gamma(0,0,w)=\partial_x\gamma(0,0,w)=\partial_y\gamma(0,0,w)=0.\end{cases}\tag{3}$$

This system of coordinates is unique up to an action of $SO(2)$ and adapted to the contact structure, i.e. it induces a gradation with respective weights $(1,1,2)$ on the space of formal power series in (x,y,w).

The truncation at the order $(-1,-1)$ of this normal form is precisely the usual left-invariant metric on the Heisenberg group, i.e. $(X_1,X_2)=(\partial_x+\frac{y}{2}\partial_w,\partial_y-\frac{x}{2}\partial_w)$.

Given a Heisenberg geodesic with initial covector $\lambda_0=(p(0),q(0),r(0))\in T_0^*M$, the corresponding conjugate time is exactly $\tau_{\text{conj}}=2\pi/r(0)$. In the general contact case (see e.g. [5, Chapter 16]), the first conjugate time is of the form

$$\tau_{\text{conj}}=2\pi/r(0)+O(1/r(0)^3)$$

where the higher order terms can be expressed via the coefficients of the Taylor expansions

$$\beta(x,y,w)=\sum\nolimits_{l=1}^{k}\beta^l(x,y,w)+O^{k+1}(x,y,w),\qquad \gamma(x,y,w)=\sum\nolimits_{l=2}^{k}\gamma^l(x,y,w)+O^{k+1}(x,y,w)$$

with respect to the gradation $(1,1,2)$. We introduce in the following equations the coefficients (c_i,c_{jk},c_{lmn}) of the polynomial functions γ^2,γ^3 and γ^4 appearing in these expansions.

$$\begin{cases} \gamma^2(x,y)=(c_0+c_2)(x^2+y^2)+(c_0-c_2)(x^2-y^2)-2c_1xy\ ,\\ \gamma^3(x,y)=(c_{11}x+c_{12}y)(x^2+y^2)+3(c_{31}x-c_{32}y)(x^2-y^2)-2(c_{31}x^3+c_{32}y^3)\\ \gamma^4(x,y,w)=\dfrac{w}{2}\Big((c_{421}+c_{422})(x^2+y^2)+(c_{421}-c_{422})(x^2-y^2)-2c_{423}xy\Big)+c_{441}(x^2+y^2)^2\\ \qquad +c_{442}(x^4+y^4-6x^2y^2)+4c_{443}\,xy(x^2-y^2)+c_{444}(x^4-y^4)-2c_{445}\,xy(x^2+y^2)\end{cases}\tag{4}$$

These coefficients derive from the irreductible decompositions of γ^2, γ^3 and γ^4 under the action of $SO(2)$, and are precisely the fundamental invariants discriminating the singularities listed in Theorem 1.1. In particular, the decompositions of β, γ^2 and γ^3 have an interpretation in terms of canonical sub-Riemannian *connection* and *curvature*, as detailed in [6].

Following the methodology developed in [3], one introduces the coordinates (h, φ) defined by

$$(p, q) = (\cos(\varphi), \sin(\varphi)),\ \ h = \sqrt{|w|/\pi}$$

and expresses the Taylor expansion of order $k \geq 3$ of the semi conjugate loci as the suspension

$$\mathrm{Conj}_{\pm}(\varphi, h) = (x(\varphi, h), y(\varphi, h), h) = \left(\sum\nolimits_{l=3}^{k} h^l f_l^{\pm}(\varphi), h\right), \tag{5}$$

where the $\pm$ symbol highlights the dependence of the expression on the sign of w. By carrying out explicitly the computations necessary to put the conjugate locus in the suspended form (5), it can be verified that $f_3^+ = f_3^-$ and $f_4^+ = f_4^-$. Hence, the generic behaviour of the full conjugate locus is already known outside the smooth curve $\mathscr{C} \subset M$ on which $c_0 = c_2$. We therefore restrict our attention to the more degenerate singularities arising in the form of *self-intersections* along the curve $\mathscr{C}$.

To prove that the symbols listed in Theorem 1.1 are generic, one needs to compute the Taylor expansions of order $k = 7$ of the metric in a neighbourhood of the origin. All these expressions where obtained using Maple software using a piece of code that can be found at the following address:

`http://www.lsis.org/bonnetb/depots/Sub-Riemannian_Conjugate_Locus`

3. Generic singularities of the full-conjugate locus

In this section, we prove Theorem 1.1. We show that the coefficients defined in (4), characterizing the generic conjugate locus on the curve $\mathscr{C}$, generate independent structures for Conj_+ and Conj_-.

Definition 3.1 (Self-intersection set of the semi-conjugate loci) *We define the self-intersection set of* Conj_+ *as*

$$\mathrm{Self}(\mathrm{Conj}_+) = \{(h, \varphi_1, \varphi_2) \in \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1 \ s.t.\ h > 0,\ \varphi_1 \neq \varphi_2,\ \mathrm{Conj}_+(h, \varphi_1) = \mathrm{Conj}_+(h, \varphi_2)\}.$$

An angle $\varphi \in \mathbb{S}^1$ *is said to be* adherent *to* $\mathrm{Self}(\mathrm{Conj}_+)$ *provided that* $(0, \varphi, \varphi + \pi) \in \overline{\mathrm{Self}(\mathrm{Conj}_+)}$. *The set of such angles is denoted by* $\in$ $\mathrm{A\text{-}Self}(\mathrm{Conj}_+)$. *We define in the same way the self-intersection and adherent angles sets of* Conj_-.

We recall in Theorem 3.2 below the complete classification of the generic self-intersections of Conj_+ which was derived in [3].

Theorem 3.2 (Generic self-intersections of the positive semi conjugate locus) *Let* $\mathscr{C} \subset M$ *be the curve defined in Theorem 1.1. Then,* $\mathscr{C}$ *is a generically smooth curve, and outside* $\mathscr{C}$ *the semi conjugate loci are the standard 4 cusp semi-caustics. On* $\mathscr{C}$, *the following situations can occur*

- *(i) There exists an open and dense subset* $\mathscr{O}^+ \subset \mathscr{C}$ *on which the self-intersections of* Conj_+ *are described by the symbols* $(\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3)$.
- *(ii) There exists a discrete subset* $\mathscr{D}^+$ *complement of* $\mathscr{O}^+$ *in* $\mathscr{C}$ *on which the self-intersections of* Conj_+ *are described by the symbols* $(\mathscr{S}_4, \mathscr{S}_5, \mathscr{S}_6, \mathscr{S}_7)$.

We recall in the following Lemma the structural result allowing to describe the sets $\mathrm{A\text{-}Self}(\mathrm{Conj}_{\pm})$.

Lemma 1 (Structure of the self-intersection) *The set of adherent angle to* $\mathrm{Self}(\mathrm{Conj}_{\pm}$ *satisfies the inclusion* $\mathrm{A\text{-}Self}(\mathrm{Conj}_{\pm}) \subset \{\varphi \in \mathbb{S}^1 \ s.t.\ f_4'(\varphi) \wedge f_5^{\pm}(\varphi) = 0\}$, *where* $a \wedge b \equiv \det(a, b)$ *for vectors* $a, b \in \mathbb{R}^2$.

An explicit computation based on the expressions of the maps f_4, f_5^- and f_5^+ defined in (5) shows that

$$f_4'(\varphi) \wedge f_5^{\pm}(\varphi) = -20\pi^2 \tilde{b} \sin(3\varphi + \omega_b) P_{\pm}(\varphi)$$

where we introduced the notations $(c_{31}, c_{32}) = \tilde{b}(\sin(\omega_b), -\cos(\omega_b))$ and

$$P_\pm(\varphi) = A_\pm \cos(2\varphi) + B_\pm \sin(2\varphi) + C\cos(4\varphi) + D\sin(4\varphi).$$

Here, the coefficients $(A_\pm, B_\pm, C, D)$ are **independent linear combinations** of the fourth-order coefficients $(c_{421}, c_{422}, c_{423}, c_{442}, c_{443}, c_{444}, c_{445})$ introduced in (4). Their analytical expressions were derived again by using the Maple software and write as follows:

$$\begin{cases} A_\pm = \frac{35}{8}(c_{422} - c_{421}) \pm 3\pi c_{423} + 45c_{445}, \ C = 36c_{443}, \\ B_\pm = \frac{35}{8}c_{423} \pm 3\pi(c_{421} - c_{422}) + 45c_{444}, \ D = -36c_{442}. \end{cases}$$

This independence result allows to show that the transversality conditions used in the classification of the semi conjugate loci hold independently for Conj_+ and Conj_-. Hence, the obstructions to the pairing of two given symbols can be expressed in terms of the codimensions of the corresponding singularities. This independence result implies in particular the following structural corollary for the cut locus.

Corollary 2 (Generic behaviour of the 3D-contact sub-Riemannian cut locus) *There exists an open and dense subset $\mathscr{E} \subset SubR(M)$ for the Whitney topology such that for any $(\Delta, \mathbf{g}) \in \mathscr{E}$, the following holds.*

(i) Outside the smooth curve $\mathscr{C} \subset M$ defined in Theorem 1.1, the semi cut loci are independent : each of the two is a portion of plane joining two opposite plea lines along the corresponding semi caustic (see Figure 1-left).

(ii) On the curve $\mathscr{C}$, the semi cut loci are the union of three portions of planes connecting the h-axis with alternate plea lines along the corresponding semi caustic (see Figure 3-left dotted lines). There is no interdependence whatsoever between these planes for the positive and negative semi cut loci.

The generic behaviour of the 3D-contact semi cut loci outside the degenerate curve $\mathscr{C}$ is already known (see e.g. [5, Chapter 16]). As one approaches $\mathscr{C}$, swallow tails forms themselves along the semi caustics (see Figure 1-right) and the four plea lines of the caustic degenerate into the 6-cusp structure studied in this paper. The corresponding cusps are regrouped by pairs of the form $(k, k+1)$, appearing respectively in a small vicinity of the cuspidal angles $\frac{k}{3}(\pi - \omega_b)$ with $k \in \{0, 1, 2\}$.

Generically, the semi cut loci are the unions of three portions of planes joining three of the six plea lines to the h-axis. Moreover, in the absence of an interior cusp, these supporting plea lines can be chosen freely for each semi cut loci to be either one of the two numberings $(1, 3, 5)$ or $(2, 4, 6)$ of the six plea lines. The result of this paper stating that the cusps of the conjugate locus are independently distributed for its upper and lower parts therefore yields Corollary 2.

References

[1] A. Agrachev, *Methods of Control Theory in Nonholonomic Geometry.* Proceed. ICM-94, Zürich, Birkhaüser, 1995.

[2] A. Agrachev, *Exponential Mappings for Contact Sub-Riemannian Structures.* Journal of Dynamical Control Systems, Vol.2, N.3, 1996, pp. 321-358.

[3] A. Agrachev, H. Chakir J.P. Gauthier, *Subriemannian metrics on* $\mathbb{R}^3$. Geometric control and nonholonomic Mechanics, Proceedings of Canad. Math. Soc. Vol. 25, 1998, pp. 29-76.

[4] A. Agrachev, G. Charlot, J.P. Gauthier, V. Zakalyukin, *On Subriemannian Caustics and Wave Fronts for Contact Distributions in the three Space.* Journal of Dynamical and control Systems, Vol. 6, N.3, pp. 365-395, 2000.

[5] A. Agrachev, D. Barilari and U. Boscain, *A Comprehensive Introduction to Sub-Riemannian geometry.* To appear in Cambridge University Press, `https://webusers.imj-prg.fr/~davide.barilari/2017-11-17-ABB.pdf`.

[6] H. Chakir, J.P. Gauthier, I. Kupka, Small Sub-Riemannian Balls on $\mathbb{R}^3$. Journal of Dynamical and Control Systems, Vol.2, N.3, 1996, pp. 359-421.

[7] M.W.Hirsch, *Differential Topology*, Springer Graduate Texts in Mathematics, Vol. 33, 1974.