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Extensions of "Padé Discretization for Linear Systems With Polyhedral Lyapunov Functions" for generalised Jordan structures

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***Abstract*—Recently, we showed that certain types of polyhedral Lyapunov functions for linear time-invariant systems, are preserved by diagonal Padé approximations, under the assumption that the continuous-time system matrix A_c has distinct eigenvalues. In this paper we show that this result also holds true in the case that A_c has non-trivial Jordan blocks.**

***Index Terms*—Polyhedral Lyapunov functions, Preservation of Lyapunov functions, Discretization, Padé approximations, Non-trivial Jordan blocks.**

I. Introduction

RECENTLY, we showed that certain types of polyhedral Lyapunov functions for linear time-invariant systems, are preserved by diagonal Padé approximations, under the assumption that the continuous-time system matrix A_c has distinct eigenvalues [1]. This result follows by making explicit use of the fact that the diagonal Padé approximation preserves the Jordan structure of a matrix A_c if the matrix has distinct eigenvalues. Unfortunately, this fact no longer holds when A_c has non-trivial Jordan blocks, and the purpose of this paper is therefore to extend the results of [1] to the case of non-trivial Jordan blocks.

Polyhedral Lyapunov functions are known to be nonconservative in the analysis of stability under arbitrary switching for polytopic and switched systems, when compared to quadratic Lyapunov functions [2]. The motivation for wondering whether there exists a polyhedral LF that is shared under discretization is discussed in [1]. Recall that the investigation of the preservation of properties of linear systems when passing from the continuous-time analysis to the discrete-time one has been the subject of great interest in the control theory community [3]. This investigation for linear time-invariant systems is mature. On the other hand, the theory of switched linear systems on the other hand, is a relatively new field of research where the knowledge of the shared properties between continuous-time and discrete-time systems is nearly absent. In particular, in studying systems that involve switching control engineers are interested in developing discretisation methods that yield discrete-time approximations that inherit some of the qualitative properties of the original system. Stability is one such property and it is in this context that polyhedral Lyapunov functions arise; recall, the converse theorems tell us that (roughly speaking), exponential stability and polyhedral stability are interchangeable concepts even for switched linear systems [4]. The question as to which discretisation methods preserve polyhedral Lyapunov functions is thus of fundamental interest. The present paper completes the work started in [1] as it extends the given results even to the case where the system matrix has non-trivial Jordan blocks.

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Notation: The notation $\mathcal{R}$ is used to denote the set of real numbers. In this paper we consider the ∞-measure of a square matrix X defined as $\mu_\infty(X) = \max_i \left(X_{ii} + \sum_{j\neq i} |X_{ij}| \right)$ and the ∞-norm as $\|X\|_\infty = \max_i \sum_j |X_{ij}|$, where the terms X_{ij} are the entries of X.

II. Problem statement

Consider a continuous-time linear time-invariant (LTI) system

$$\dot{x}(t) = A_c x(t), \tag{1}$$

where $x \in \mathcal{R}^n$ and the matrix $A_c \in \mathcal{R}^{m\times m}$. We are interested in the discrete time approximation to this system with a positive sampling time $h \in \mathcal{R}$ given by

$$x(k+1) = A_d x(k) \tag{2}$$

where A_d is obtained via the diagonal Padé approximation to $e^{A_c h}$.

Definition 1: [8] Let n be a positive integer, then the n^{th} order diagonal Padé approximation to the exponential function e^z is the rational function $P_{[n/n]}$ defined by $P_{[n/n]}(z) = \frac{Z_n(z)}{Z_n(-z)}$ where

$$Z_n(z) = \sum_{k=0}^{n} c_k z^i \quad \text{and} \quad c_i = \frac{(2n-i)!n!}{(2n)!i!(n-i)!}. \tag{3}$$

Thus the n^{th} order diagonal Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h, is given by

$$A_d = P_{[n/n]}(A_c h) = Z_n(A_c h) Z_n^{-1}(-A_c h) \tag{4}$$

where $Z_n(A_c h) = \sum_{k=0}^{n} c_k (A_c h)^k$. An important property of diagonal Padé approximation is that, roots of the polynomial $Z_n(-z)$ (poles of $P_{[n/n]}(z)$) lie in the open right half plane [8]. Thus $P_{[n/n]}(z)$ is analytic on the closed left half plane. Furthermore, diagonal Padé approximations map the closed left half plane into the closed unit circle, hence the eigenvalues of A_d lie inside the unit circle [8] whenever A_c is Hurwitz. We assume that all the continuous time system matrices A_c

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are Hurwitz. Consider now a piecewise linear (polyhedral), candidate Lyapunov function $V(x)$ given by

$$V(x) = \|Wx\|_\infty \tag{5}$$

where $W \in \mathcal{R}^{N\times m}$ is full rank matrix and x is state vector given by equations (1) or (2).

Definition 2: [5] Consider $V(x)$ given by (5), then

(i) $V(x)$ is polyhedral Lyapunov function for (1) if and only if

$$\lim_{\tau\to 0^+} \frac{\|W(I+\tau A_c)x\|_\infty - \|Wx\|_\infty}{\tau} < 0 \text{ for all } x \neq 0. \tag{6}$$

(ii) $V(x)$ is polyhedral Lyapunov function for (2) if and only if

$$\|WA_d x\|_\infty - \|Wx\|_\infty < 0 \text{ for all } x \neq 0. \tag{7}$$

Now we present necessary and sufficient conditions under which conditions (6) and (7) are satisfied for functions $\|W_c x\|_\infty$ and $\|W_d x\|_\infty$ respectively [5].

Lemma 3: Given a full column rank matrix $W_c \in \mathcal{R}^{N\times m}$, $N \geq m$, the function $V_c(x) := \|W_c x\|_\infty$ is a Lyapunov function for the continuous-time system $\dot{x} = A_c x$ if there exists $Q_c \in \mathcal{R}^{N\times N}$ such that

$$W_c A_c = Q_c W_c, \quad \mu_\infty(Q_c) < 0.$$

Lemma 4: Given a full column rank matrix $W_d \in \mathcal{R}^{N\times m}$, $N \geq m$, the function $V_d(x) := \|W_d x\|_\infty$ is a Lyapunov function for the discrete-time system $x(k+1) = A_d x(k)$ if there exists $Q_d \in \mathcal{R}^{N\times N}$ such that

$$W_d A_d = Q_d W_d, \quad \|Q_d\|_\infty < 1.$$

In this paper we are interested in the preservation of polyhedral Lyapunov functions under diagonal Padé approximation. Recall in [1], we have proved the following fundamental result:

Theorem 5: Consider a Hurwitz stable matrix A_c of dimension N and its diagonal Padé discretization A_d of order n. **Assume that all eigenvalues of A_c are distinct**. Let N_r be the number of real negative eigenvalues, and $2N_c$ be the number of pairs of complex eigenvalues $\sigma_i \pm j\tau_i$, $i = 1, 2, \cdots, N_c$. For each pair of complex eigenvalues, let k_i be an integer greater than one such that $\sigma_i \pm j\tau_i$ belongs to the sector $\mathcal{S}_c(k_i) := \{\lambda = \sigma + j\tau : \quad \sigma < 0, \ |\tau| < \frac{sin(\frac{\pi}{k_i})}{1-cos(\frac{\pi}{k_i})}|\sigma|\}$. Then there exist $W \in \mathcal{R}^{N'\times N}$, with $N' = \sum_{i=1}^{k} k_i + N_r$ with

$$W = \begin{pmatrix} W_1 & 0 & \cdots & 0 & 0 \\ 0 & W_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{N_c} & 0 \\ 0 & 0 & \cdots & 0 & I \end{pmatrix} T_c, \tag{8}$$

$$\text{where } W_i = \begin{pmatrix} 1 & 0 \\ cos(\frac{\pi}{k_i}) & sin(\frac{\pi}{k_i}) \\ cos(\frac{2\pi}{k_i}) & sin(\frac{2\pi}{k_i}) \\ \vdots & \vdots \\ cos(\frac{(k_i-1)\pi}{k_i}) & sin(\frac{(k_i-1)\pi}{k_i}) \end{pmatrix},$$

and T_c is the Modal matrix for A_c, such that $V(x) := \|Wx\|_\infty$ is a Lyapunov function both for A_c and A_d.

The geometrical meaning of sectors $\mathcal{S}_c(k)$ is explained in detail in [1]. In this article, we prove the same result when A_c has non-trivial Jordan blocks. Given A_c a square matrix, we consider its real Jordan form $J_c = T_c^{-1} A_c T_c$:

$$J_c = \begin{pmatrix} J_c^0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & J_c^1 & \dots & \mathbf{0} \\ \vdots & & & \\ \mathbf{0} & \dots & \mathbf{0} & J_c^l \end{pmatrix}, \tag{9}$$

where $J_c^1, \dots, J_c^l$ are all the blocks either of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \tag{10}$$

with $\lambda < 0$ (real eigenvalues), or of the form:

$$\begin{pmatrix} \Lambda & I & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Lambda & I & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Lambda & I \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \Lambda \end{pmatrix} \tag{11}$$

where $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$, $\sigma < 0$, $\tau > 0$, I is the identity matrix of dimension 2 and $\mathbf{0}$ is the null matrix of dimension 2. The first block J_c^0 has the following structure

$$J_c^0 = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 \ \ 0 & \dots & 0 \ \ 0 \\ \vdots & & & & & \\ 0 & \dots & \lambda_{n_0} & 0 \ \ 0 & \dots & 0 \ \ 0 \\ 0 & \dots & 0 & \Lambda_1 & \dots & 0 \ \ 0 \\ 0 & & 0 & & & 0 \ \ 0 \\ \vdots & & & & & \\ 0 & \dots & 0 & 0 \ \ 0 & \dots & \Lambda_{m_0} \\ 0 & & 0 & 0 \ \ 0 & & \end{pmatrix}$$

with $\Lambda_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}$. In other words, J_c^0 contains the real eigenvalues λ_i (eventually coinciding) such that the corresponding line and column in the real Jordan form are 0 except on the diagonal itself, and the complex eigenvalues blocks Λ_i such that the corresponding lines and columns are 0 except on the block itself.

We now state our main result:

Theorem 6: Let A_c be a Hurwitz matrix. Then, there exists a matrix W such that for all $h > 0$ and any order n of approximation, the systems (1) and (2) with $A_d = P_{[n/n]}(A_c h)$ share the polyhedral Lyapunov function $\|Wx\|_\infty$. We have $W = \tilde{W} T_c$ where T_c is the modal matrix for A_c and the precise structure of $\tilde{W}$ is given in Lemmas 7 and 9.

We shall prove this theorem in Section V. The proof is based on the study of each block of the matrix J_c, the real-Jordan form for A_c. For this reason, we first study the two following special cases: the **real case**, in which A_c is given by (10); and the **complex case**, in which A_c is given (11). These cases are presented in the two following sections.

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III. The real case

In this section, we consider A_c of the form (10). We denote its dimension with m. Then

$$A_d = P_{[n/n]}(A_c h) = \begin{pmatrix} f_0 & f_1 & f_2 & \dots & f_{m-1} \\ 0 & f_0 & f_1 & \dots & f_{m-2} \\ 0 & 0 & f_0 & \dots & f_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_0 \end{pmatrix}, \quad (12)$$

with $f_i := P^{(i)}_{[n/n]}(\lambda h)\frac{h^i}{i!}$. The index $^{(i)}$ denotes the i-th derivative of $P_{[n/n]}(x)$ with respect to x. This formula can be easily proved by writing $P_{[n/n]}(x) = \sum_{i=0}^{\infty} a_i x^i$ and studying the expression of the powers A_c^i; see [7] for details. As a consequence, terms on the upper diagonal have series expressions coinciding with derivatives of the series $\sum_{i=0}^{\infty} a_i x^i$. The convergence of the series for A_c Hurwitz is given by the fact that Padé approximation and its derivatives have poles in the open right-half plane only [8].The formula (12) indicates the primary motivation for extending the results obtained in [1]. It can be observed that the function of a matrix with non-trivial Jordan blocks is not in its Jordan form, hence the modal matrices for A_c and $P_{[n/n]}(A_c h)$ are not the same. However, the results in [1] are based on the assumption that A_c has distinct eigenvalues leading to the same modal matrices for A_c and $P_{[n/n]}(A_c h)$. This observation motivates this paper to extend the results from [1] for a more general case with matrix A_c having non-trivial Jordan blocks. We now prove the following lemma.

Lemma 7: Consider the Hurwitz matrix A_c of the form (10) and denote its dimension with m. Then, there exists a positive $\alpha > -\frac{1}{\lambda}$ such that for all $h > 0$ and any order n of approximation, the matrices A_c and $A_d = P_{[n/n]}(A_c h)$ share the common Lyapunov function

$$V(x) = \|Dx\|_\infty. \quad (13)$$

with $D = \text{diag}\{1, \alpha \dots, \alpha^{m-1}\}$.

Proof : We first prove that (13) is a Lyapunov function for A_c using the conditions from Lemma 3. Since D is invertible, we transform $DA_c = Q_c D$ to $Q_c = DA_cD^{-1} = \begin{pmatrix} \lambda & \frac{1}{\alpha} & 0 & \dots & 0 \\ 0 & \lambda & \frac{1}{\alpha} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & \frac{1}{\alpha} \\ 0 & 0 & \dots & \dots & \lambda \end{pmatrix}$. Under the condition $\alpha > -\frac{1}{\lambda}$, we have that $\mu_\infty(Q_c) = \lambda + \frac{1}{\alpha} < 0$. Thus $V(x) = \|Dx\|_\infty$ is a Lyapunov function for A_c. Since $A_d = P_{[n/n]}(A_c h)$ is given by (12), we can compute $Q_d = DP_{[n/n]}(A_c h)D^{-1}$, that is $Q_d = \begin{pmatrix} f_0 & \frac{f_1}{\alpha} & \frac{f_2}{\alpha^2} & \dots & \frac{f_{m-1}}{\alpha^{m-1}} \\ 0 & f_0 & \frac{f_1}{\alpha} & \dots & \frac{f_{m-2}}{\alpha^{m-2}} \\ 0 & 0 & f_0 & \dots & \frac{f_{m-3}}{\alpha^{m-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_0 \end{pmatrix}$. To show that (13) is a Lyapunov function for A_d we need to prove $\|Q_d\|_\infty < 1$ (see Lemma 4), which is further equivalent to prove that

$$|f_0| + \frac{|f_1|}{\alpha} + \frac{|f_2|}{\alpha^2} + \dots + \frac{|f_{m-1}|}{\alpha^{m-1}} < 1 \quad (14)$$

Take now any $\bar{h} > 0$. We compute α such that (14) is satisfied for all $h < \bar{h}$. For each $i = 1, \dots, m-1$, compute $M^i = \max_{h \in [0,\bar{h}]} |P^{(i)}_{[n/n]}(\lambda h)|$ and $M = \max_{i=1,\dots,m-1} M^i$. Remark that each M^i is finite, since the derivatives $P^{(i)}_{[n/n]}$ are always finite for non-positive numbers, since Padé approximations and their derivatives have poles with real part that is strictly positive. Hence, M exists, since it is the maximum over a finite set. Then, we bound (14) with $|P_{[n/n]}(\lambda h)| + M\left(\frac{h}{\alpha} + \dots + \frac{h^{m-1}}{(m-1)!\alpha^{m-1}}\right) < 1$. Since $\lambda < 0$, we have $|P_{[n/n]}(\lambda h)| < 1$, hence we can always find α such that $\left(\frac{\bar{h}}{\alpha} + \dots + \frac{\bar{h}^{m-1}}{(m-1)!\alpha^{m-1}}\right) < \frac{1-|P_{[n/n]}(\lambda h)|}{M}$. This latter fact follows from the fact that $|P_{[n/n]}(\lambda h)|$ is always bounded away from 1. Thus, for all $h < \bar{h}$, condition (14) are verified.

We now study the limiting case as $h \to \infty$. First, define the new variable[1] $x := -\frac{1}{\lambda h}$ and the function $g_0(x) := f_0 = P_{[n/n]}(-1/x)$, that is defined for $x \geq 0$. In particular, at $x = 0$ we have $g_0(0) = \lim_{h\to\infty} f_0 = \pm 1$, since $|P_{[n/n]}(\infty)| = 1$. Moreover, $|P_{[n/n]}| < 1$ for all $h > 0$, that implies $|g_0(x)| < 1$ for $x > 0$. Its Taylor expansion in 0 (for $x > 0$ only) is thus $g_0(x) = d_0 + d_1 x + o(x)$ with $|d_0| = 1$ and[2] $d_0 d_1 < 0$. By substitution, we have $f_0 = d_0 + \frac{d_1}{\lambda h} + o(1/h)$. Differentiating this series i times with respect to h, we have

$$P^{(i)}_{[n/n]}(\lambda h) = (-1)^i \frac{d_1 i!}{\lambda^{i+1} h^{i+1}} + o(1/h^{i+1}),$$

thus $|f_i| = \frac{|d_1|}{|\lambda|^{i+1} h} + o(1/h)$. Since $\beta = |\lambda|\alpha > 1$, we have $\frac{|f_i|}{\alpha^i} = \frac{|d_1|}{\beta^{i-1}|\lambda|^2 \alpha h} + o(1/h) \leq \frac{|d_1|}{|\lambda|^2 \alpha h} + o(1/h)$ for all $i \geq 1$. Thus

$$\begin{aligned} \|Q_d\|_\infty &= |f_0| + \frac{|f_1|}{\alpha} + \frac{|f_2|}{\alpha^2} + \dots + \frac{|f_{m-1}|}{\alpha^{m-1}} \\ &\leq 1 - \frac{|d_1|}{|\lambda| h} + \frac{|d_1|}{|\lambda|^2 \alpha h}(m-1) + o(1/h). \end{aligned}$$

Take $\alpha > \frac{m-1}{|\lambda|}$. Then one has $\|Q_d\|_\infty \leq 1 - \frac{\delta}{h} + o(1/h)$ with δ positive. Take now h^* such that $|o(1/h)| < \delta/(2h)$ for all $h > h^*$, that exists by definition of $o(1/h)$. Then, for all $h > h^*$ one has $\|Q_d\|_\infty < 1 - \frac{\delta}{2h} < 1$.

We now merge the two cases. First use the limit case: take $\alpha_1 > \frac{m-1}{|\lambda|}$ and the corresponding h^* so that $\|Q_d\|_\infty < 1$ holds for all $h > h^*$. Use now the first part of the proof: choose any $\bar{h} > h^*$ and find the corresponding α_2 such that

[1] This process coincides with the Taylor expansion of $P_{[n/n]}$ at ∞: given a function $f(x)$, define the function $g(x) := f(1/x)$ and compute its 1^{st} order Taylor expansion at 0, that is $g(x) = a_0 + a_1 x + o(x)$. By inverting such formula, one has $f(x) = b_0 + b_1/x + o(1/x)$, that describes the behavior of $f(x)$ around $x = \infty$. The same idea can be used for higher order Taylor expansions.

[2] For $d_0 = 1$, one needs $d_1 < 0$ to have g decreasing; and the opposite for $d_0 = -1$. More precisely, g_0 decreasing with $d_0 = 1$ implies $d_1 < 0$ or $d_1 = d_2 = 0$ and $d_3 < 0$, or $d_1 = d_2 = d_3 = d_4 = 0$ and $d_5 < 0$,... For simplicity of notation, we study the case $d_1 \neq 0$; the same proof can be adapted to the other cases too.

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(14) holds for all $h < \bar{h}$. Finally choose $\alpha^* = \max\{\alpha_1, \alpha_2\}$ and observe that (14) holds for all h. □

Remark **8:** As apparent from the proof of the previous theorem, there exists a value $\bar{\alpha}$ of α that ensures that $V(x) = \|Dx\|_\infty$ is a Lyapunov function for both A_c in (10) and A_d in (12), for all $\alpha > \bar{\alpha}$. Taking again into account the fact that $\alpha|\lambda| > 1$, an upper bound value of $\bar{\alpha}$ can be found from (14), i.e.

$$\frac{1}{|\lambda|} < \bar{\alpha} \leq \sup_{h>0} \frac{\sum_{i=1}^{m-1} |f_i||\lambda|^{i-1}}{(1-|f_0|)}$$

IV. The complex case

In this section, we consider A_c of the form (11). We denote its dimension with $2m$. Then

$$A_d = P_{[n/n]}(A_c h) = \begin{pmatrix} F_0 & F_1 & F_2 & \dots & F_{m-1} \\ 0 & F_0 & F_1 & \dots & F_{m-2} \\ 0 & 0 & F_0 & \dots & F_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_0 \end{pmatrix}, \quad (15)$$

with $F_i := P^{(i)}_{[n/n]}(\Lambda h)\frac{h^i}{i!}$. As in the previous Section, derivative notation should be interpreted as the rational functions that are derivatives of the rational function $P_{[n/n]}(x)$. The proof of this formula is as for the real case. The only detail to be careful with is that, in this case, the product of matrices only involves Λ and I, for which the product is commutative.

Let k be a natural number such that $\sigma + j\tau \in \mathcal{S}_c(k)$ and let $\tilde{W} = \begin{pmatrix} 1 & 0 \\ cos(\frac{\pi}{k}) & sin(\frac{\pi}{k}) \\ cos(\frac{2\pi}{k}) & sin(\frac{2\pi}{k}) \\ \vdots & \vdots \\ cos(\frac{(k-1)\pi}{k}) & sin(\frac{(k-1)\pi}{k}) \end{pmatrix}$. This matrix defines a Lyapunov function $\|\tilde{W}x\|_\infty$ both for the block $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$ and $P_{[n/n]}(\Lambda h)$ for all $h > 0$, as proved in [5]. We now use this fact to compute the Lyapunov function for A_c and A_d, and consequently prove the following lemma.

Lemma 9: Consider the Hurwitz matrix A_c of the form (11) and denote its dimension with $2m$. Then there exists an $\alpha > \frac{1}{-\sigma - \tau\frac{1-cos(\frac{\pi}{k})}{sin(\frac{\pi}{k})}}$ such that for all $h > 0$ and order n of approximation, the matrices A_c and $A_d = P_{[n/n]}(A_c h)$ share the common Lyapunov function

$$V(x) = \|Wx\|_\infty \quad (16)$$

with $W = \begin{pmatrix} \tilde{W} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \tilde{W}\alpha & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{W}\alpha^2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \tilde{W}\alpha^{m-1} \end{pmatrix}$.

Proof : Using the conditions from Lemma 3 we first prove that (16) is a Lyapunov function for A_c. We already know that there exists a certain $\tilde{Q}_c$ with $\mu_\infty(\tilde{Q}_c) < 0$ satisfying $\tilde{W}\Lambda = \tilde{Q}_c\tilde{W}$. Moreover, $\mu_\infty(\tilde{Q}_c) = |\sigma| - \frac{|\tau|cos(\frac{\pi}{k})}{sin(\frac{\pi}{k})} + \frac{|\tau|}{sin(\frac{\pi}{k})} < 0$. See details in [1], [5]. Thus $WA_c = Q_cW$ is satisfied, with

$$Q_c = \begin{pmatrix} \tilde{Q}_c & I/\alpha & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \tilde{Q}_c & I/\alpha & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \tilde{Q}_c \end{pmatrix}$$

We have $\mu_\infty(Q_c) = \mu_\infty(\tilde{Q}_c) + \frac{1}{\alpha} < 0$ due to the condition on α. Remark that such α exists, since $\sigma + j\tau \in \mathcal{S}_c(k)$ is equivalent to $\frac{1}{-\sigma-\tau\frac{1-cos(\frac{\pi}{k})}{sin(\frac{\pi}{k})}} > 0$. Thus $V(x) = \|Wx\|_\infty$ is a Lyapunov function for A_c.

Compute now $A_d = P_{[n/n]}(A_c h)$, that is given by (15). We have to find Q_d satisfying $WA_d = Q_dW$ and $\|Q_d\|_\infty < 1$ (see Lemma 4). As a candidate, we look for

$$Q_d := \begin{pmatrix} Q_0 & Q_1/\alpha & Q_2/\alpha^2 & \dots & Q_{m-1}/\alpha^{m-1} \\ \mathbf{0} & Q_0 & Q_1/\alpha & \dots & Q_{m-2}/\alpha^{m-2} \\ \mathbf{0} & \mathbf{0} & Q_0 & \dots & Q_{m-3}/\alpha^{m-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_0 \end{pmatrix}$$

with $Q_0, Q_1, \dots Q_{m-1}$ to be found. The explicit computation of $WA_d = Q_dW$ gives the following conditions

$$\tilde{W}F_0 = Q_0\tilde{W}, \quad \tilde{W}F_i = Q_i\tilde{W}, \quad i = 1, \dots, m-1 \quad (17)$$

Since $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$, then the eigenvalues of $F_0 = P_{[n/n]}(\Lambda h)$ lie in $\mathcal{P}_{ol}(k) := int\ conv\left\{e^{j\frac{p\pi}{m}}\right\}_{p=0}^{2m-1}$, as we proved in [1]. As a consequence, for each $h > 0$ there exists Q_0 such that $\tilde{W}F_0 = Q_0\tilde{W}$ and $\|Q_0\|_\infty < 1$, see [6].

For each other F_i, observe that its entries are all bounded functions of $h > 0$, and consequently its eigenvalues are bounded too. Thus, one can choose a $\rho > 1$ sufficiently big to have the eigenvalues of $\frac{F_i}{\rho_i}$ as small as wished. In particular, one can always have the eigenvalues of $\frac{F_i}{\rho_i}$ with norm less than R_k, the radius of a ball centered in 0 and completely contained in $\mathcal{P}_{ol}(k)$. As a consequence, there exists $\tilde{Q}_i$ satisfying $\tilde{W}\frac{F_i}{\rho_i} = \tilde{Q}_i\tilde{W}$ and $\|\tilde{Q}_i\|_\infty < 1$; see again [6]. Then the conditions in (17) are all verified by taking $Q_i = \tilde{Q}_i\rho_i$. Hence, recalling that $\alpha > \frac{1}{\mu_\infty(Q_c)}$ we have

$$\begin{aligned} \|Q_d\|_\infty &\leq \|Q_0\|_\infty + \|Q_1/\alpha\|_\infty + \dots + \|Q_{m-1}/\alpha^{m-1}\|_\infty \\ &\leq \|Q_0\|_\infty + \frac{1}{\alpha}\sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1} \end{aligned}$$

Similarly to the real case, one has to study the limit case $h \to \infty$. By developing the ∞-norm of the Q_i around ∞, one finds expressions similar to f_i in the real case, and the result follows. Notice in fact that $\|Q_0\|_\infty$, as a function of $h > 0$, can be written as $\|Q_0\|_\infty = 1 - \phi(h)$ with ϕ a strictly positive function of $h > 0$. In conclusion $\|Q_d\|_\infty < 1$ if $\alpha > \sup_{h>0} \frac{\sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1}}{1-\|Q_0\|_\infty}$. □

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Remark **10:** Also for the case of multiple complex eigenvalues, we can conclude that there exists a value $\bar{\alpha}$ of α that ensures that $V(x) = \|Wx\|_\infty$ is a Lyapunov function for both A_c in (11) and A_d in (15), for all $\alpha > \bar{\alpha}$. An upper bound value of $\bar{\alpha}$ can be found computed in the following way, i.e.

$$\frac{1}{|\mu_\infty(Q_c)|} < \bar{\alpha} \leq \sup_{h>0} \frac{\sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1}}{1 - \|Q_0\|_\infty}$$

V. Proof of Theorem 6

In this section, we now prove Theorem 6. We use Lemmas 7 and 9, as well as our result in the previous paper, Theorem 5. The basic idea is to show that we can deal with each Jordan block independently.

Take A_c a Hurwitz matrix, and $J_c = T_c^{-1} A_c T_c$ its real Jordan form (9). The fundamental observation for the following is that $A_d = P_{[n/n]}(A_c h) = T_c^{-1} P_{[n/n]}(J_c h) T_c$ with $P_{[n/n]}(J_c h) =$

$$\begin{pmatrix} P_{[n/n]}(J_c^0 h) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & P_{[n/n]}(J_c^1 h) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & P_{[n/n]}(J_c^l h) \end{pmatrix}.$$

This is a standard property of the Padé approximation, since it is a rational function of matrices. As already remarked, $P_{[n/n]}(J_c h)$ is not the real Jordan form of A_d, since $P_{[n/n]}(J_c^i h)$ are not real or complex blocks for $i > 0$. We now define W, Q_c, Q_d satisfying

$$WA_c = Q_c W, \quad WA_d = Q_d W, \tag{18}$$
$$\mu_\infty(Q_c) < 0, \quad \|Q_d\|_\infty < 1, \tag{19}$$

that ensures that $V(x) = \|Wx\|_\infty$ is a Lyapunov function both for (1) and (2) with $A_d = P_{[n/n]}(A_c h)$. First of all, we find W^i, Q_c^i, Q_d^i for each J_c^i. For the block J_c^0, use Theorem 5, that gives W^0 and the corresponding Q_c^0, Q_d^0. For blocks J_c^i, either use Lemma 7 for the real case or Lemma 9 for the complex case, that give W^i and the corresponding Q_c^i and Q_d^i.

Define $\tilde{W} = \begin{pmatrix} W^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & W^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & W^l \end{pmatrix}$ and $W = \tilde{W} T_c$.
We prove that W defines a Lyapunov function $V(x) = \|Wx\|_\infty$ both for (1) and (2) with $A_d = P_{[n/n]}(A_c h)$. It is sufficient to find Q_c and Q_d satisfying (18)-(19). By direct computation, one can prove that

$$Q_c = \begin{pmatrix} Q_c^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Q_c^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_c^l \end{pmatrix} \tag{20}$$

and $Q_d = \begin{pmatrix} Q_d^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Q_d^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_d^l \end{pmatrix}$

satisfy these conditions, since $WA_c = \tilde{W} T_c A_c = \tilde{W} J_c T_c$
$= \begin{pmatrix} W^0 J_c^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & W^1 J_c^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & W^l J_c^l \end{pmatrix} T_c = Q_c \tilde{W} T_c = Q_c W.$
The same holds for WA_d. Moreover, $\mu_\infty(Q_c) = \max_{i=0,\dots,l} \mu_\infty(Q_c^i) < 0$ and $\|Q_d\|_\infty = \max_{i=0,\dots,l} \|Q_d^i\|_\infty < 1$.

VI. Numerical Examples

Example 1: We now illustrate the result indicated in Lemma 7 using a numerical example. In particular, we show by construction the existence of a Lyapunov function that is preserved by diagonal Padé approximations of any step size and order. To this end, consider a Hurwitz matrix A_c of the form (10) with $\lambda = -3$ and $m = 3$. Then, it is easily verified that a Lyapunov function for the continuous time matrix A_c given by $V(x) = \|Dx\|_\infty$. with $D = \text{diag}\{1, \alpha, \alpha^2\}$ and $\alpha > \frac{1}{3}$. Now we consider 1^{st} order diagonal Padé approximation $A_d = P_{[1/1]}(A_c h)$ for $e^{A_c h}$ and plot the values of h and α (using '*') where $\|Q_d\|_\infty = \|DA_d D^{-1}\|_\infty > 1$. It can observed from the Figure 1 that there exists a finite limiting value of α, defining the boundary of the infeasible values of α as $h \to \infty$. We denote this value of α as $\bar{\alpha}$ and any Lyapunov function $V(x) = \|Dx\|_\infty$ with $\alpha > \bar{\alpha}$ will be preserved during discretization using diagonal Padé approximation with any step size h and order n. A similar

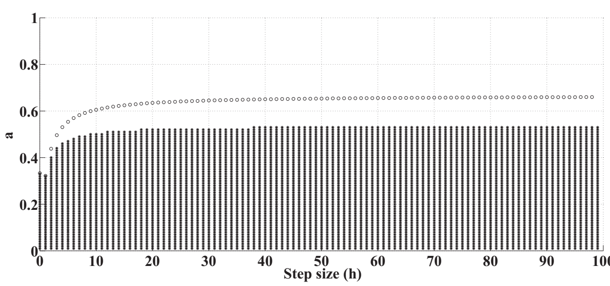

Fig. 1: Plot showing values of h and α and $L(\lambda)$

bound was proposed in Remark 8. To compare these two bounds, we plot $L(\lambda) = \frac{\sum_{i=1}^{m-1} |f_i||\lambda|^{i-1}}{(1-|f_0|)}$ w.r.t h (using 'o') in Figure 1. It can observed that the bound on $\bar{\alpha}$, proposed in Remark 8 is accurate but clearly more conservative.

Example 2: In some situations it is of interest to first define the Lyapunov function by fixing α. In such situations the pertinent problem then becomes one of estimating a minimum $\bar{h}$ for preserving the Lyapunov function. We now show how this can be achieved for matrices with real Jordan blocks using 1^{st} order diagonal Padé approximations. Consider a Hurwitz matrix A_c and the Lyapunov function $V(x)$ as defined in Example 1. Let us choose $\alpha = \alpha^* = 0.34 < \bar{\alpha}$ (from Example 1 $\bar{\alpha}$ can be approximately estimated as 0.53). If goal of discretization is to preserve this given Lyapunov function, then we need to find values of h such that $\|Q_d(\alpha^*, h)\|_\infty < 1$.

Hence we plot $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. h in Figure 2. It can be observed that $\|Q_d(\alpha^*, h)\|_\infty$ decreases monotonically for a certain range of step sizes $(0, \bar{h})$ and then starts to increase again. Our goal is to numerically evaluate this upper bound $\bar{h}$, which guarantees the preservation of Lyapunov function if $h < \bar{h}$. Note that, while this can always be done numerically,

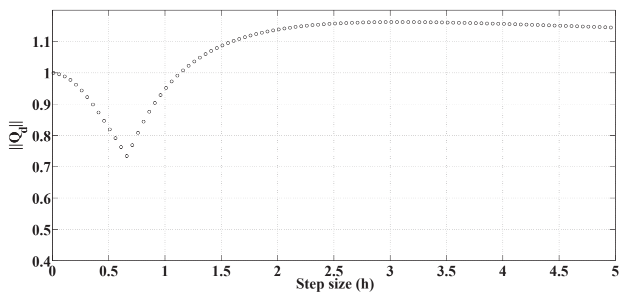

Fig. 2: Plot showing $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. h.

sometimes we can find an algebraic bound on h. To see this, consider $\|Q_d(\alpha^*, h)\|_\infty$

$$= \|DP_{[1/1]}(A_c h)D^{-1}\|_\infty = \sum_{j=0}^{2} \left| \frac{P_{[1/1]}^{(j)}(\lambda h)}{j!} \left(\frac{h}{\alpha^*}\right)^j \right| \qquad (21)$$

In the case of odd-ordered Padé approximations, we know that $P_{[n/n]}(x)$ is absolutely monotonic for $x \in (-r_n, 0]$, for a certain r_n depending on n. We recall that absolute monotonicity means that all derivatives are positive. For $P_{[1/1]}(x)$ we know that $r_1 = 2$ (see e.g. [9]), hence if we choose h such that $\lambda h > -2$, then the series (21) has all positive terms and then we can estimate (21) with $\sum_{j=0}^{\infty} \frac{P_{[1/1]}^{j}(\lambda h)}{j!} \left(\frac{h}{\alpha^*}\right)^j$

$$= P_{[1/1]}\left(\lambda_1 h + \frac{h}{\alpha^*}\right) \le |P_{[1/1]}\left(\lambda h + \frac{h}{\alpha^*}\right)|.$$

Since $\lambda h + \frac{h}{\alpha^*} < 0$ for our choice of α, we have $|P_{[1/1]}\left(\lambda h + \frac{h}{\alpha^*}\right)| < 1$. Hence $\bar{h} < \frac{r_1}{|\lambda|} = 2/3$. Some values of r_n, as well as an algorithm for their computation, are given in [9].

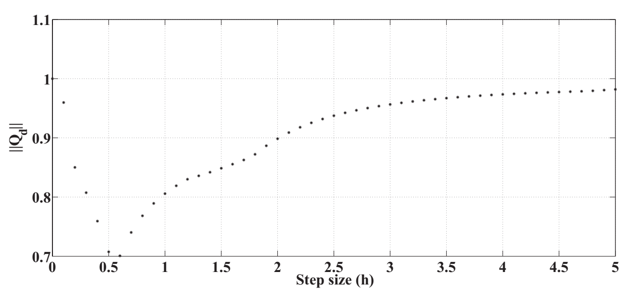

Fig. 3: Plot showing $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. $h \in [0, 5]$.

Example 3 (Complex case): Consider a Hurwitz matrix A_c of the form (11) with $\sigma = -2$, $\tau = 3$ and $2m = 4$. Since $\tau < \frac{sin(\frac{\pi}{k})}{1-cos(\frac{\pi}{k})}|\sigma|$ is verified for $k = 3$, it is easily verified that

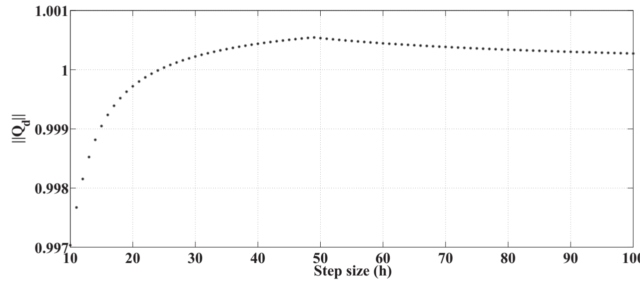

Fig. 4: Plot showing $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. $h \in [10, 100]$.

$V(x) = \|Wx\|_\infty$ is a Lyapunov function for the continuous time matrix A_c with $\tilde{W} = \begin{pmatrix} 1 & 0 \\ cos(\frac{\pi}{3}) & sin(\frac{\pi}{3}) \\ cos(\frac{2\pi}{3}) & sin(\frac{2\pi}{3}) \end{pmatrix}$ and $\alpha > \frac{1}{-2-3\frac{1-cos(\frac{\pi}{3})}{sin(\frac{\pi}{3})}} = 3.7321$ as defined in (16). Let us choose $\alpha^* = 4$. If goal of discretization is to preserve this given Lyapunov function, then we need to find values of h such that $\|Q_d(\alpha^*, h)\|_\infty < 1$. Hence we plot $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. h in Figures 3, 4. From Figure 3, it can be observed that $\|Q_d(\alpha^*, h)\|_\infty$ decreases monotonically for a certain range of step sizes $(0, \bar{h})$ and from Figures 3, 4, it can observed that $\|Q_d(\alpha^*, h)\|_\infty$ starts to increase again and crosses unity. For the complex case, $\bar{h}$ can be evaluated in a similar manner as in the real case as $\bar{h} < r_n \frac{1-cos\left(\frac{\pi}{k}\right)}{2|\sigma|} = 0.25$. It should also be noted from Figure 3, that the algebraically calculated bound $\bar{h}$ is a conservative approximation.

VII. Conclusion

In this paper we have shown that our previous results on polyhedral Lyapunov functions [1] extend to the case of linear systems with non-trivial Jordan structures. Future work will consider Padé discretisations and polynomial Lyapunov functions.

References

[1] F. Rossi, P. Colaneri, R. Shorten, Padé discretization for systems with piecewise linear Lyapunov functions, *IEEE Transactions on Automatic Control*, Vol. 56, No. 11, pp. 2717–2722, 2011.

[2] F. Blanchini and S. Miani, Set theoretic methods in control, Birkhauser, Boston, 2008.

[3] A. Zappavigna, P. Colaneri, S. Kirkland, R. Shorten, Essentially Negative News about Non-negative Matrices, Linear Algebra and its Applications, *Linear Algebra and its Applications*, Vol. 436, No. 9, pp. 3425–3442, 2012.

[4] A. Pietrus and V. Veliov, On the discretization of switched linear systems, *Systems and Control Letters*, Vol. 58, No. 6, pp. 395-399, 2009.

[5] A. Polanski, On infinity norms as Lyapunov functions for linear systems, *IEEE Transaction on Automatic Control*, Vol. 40, No. 7, pp. 1270–1274, 1995.

[6] F.K. Christophersen, M. Morari, Further results on 'Infinity norms as Lyapunov functions for linear systems', *IEEE Transactions on Automatic Control*, Vol. 52, No. 3, pp. 547-553, 2007.

[7] P. Lancaster, M. Tismenetsky, The theory of matrices: with applications, *Academic Pr*,1985.

[8] G. Birkhoff, R. S. Varga, Discretization errors for well-set Cauchy problems, *Journal of Mathematics and Physics*, Vol.44, No.1, pp. 28-51, 1965.

[9] J. A. van de Griend and J. F. B. M. Kraaijevanger, Absolute monotonicity of rational functions occurring in the numerical solution of initial value problems, *Numer. Math.*, Vol. 49, No. 4, pp. 413–424, 1986.

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