APPROXIMATE AND EXACT CONTROLLABILITY OF THE CONTINUITY EQUATION WITH A LOCALIZED VECTOR FIELD*

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Abstract. We study controllability of a Partial Differential Equation of transport type, that arises in crowd models. We are interested in controlling it with a control being a vector field, representing a perturbation of the velocity, localized on a fixed control set. We prove that, for each initial and final configuration, one can steer approximately one to another with Lipschitz controls when the uncontrolled dynamics allows to cross the control set. We also show that the exact controllability only holds for controls with less regularity, for which one may lose uniqueness of the associated solution.

Key words. Controllability, transport PDEs, optimal transportation

AMS subject classifications. 93B05; 35Q93

1. Introduction.

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community (see *e.g.* the Cucker-Smale model [22]). A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [6, 7, 12, 21, 30, 31, 36, 40].

Beside the description of interactions, it is now relevant to study problems of **control of crowds**, *i.e.* of controlling such systems by acting on few agents, or on the crowd localized in a small subset of the configuration space. The nature of the control problem relies on the model used to describe the crowd. Two main classes are widely used.

In **microscopic models**, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. For control of such models, a large literature is available from the control community, under the generic name of networked control (see *e.g.* [11, 32, 33]). There are several control applications to pedestrian crowds [26, 34] and road traffic [13, 29].

In **macroscopic models**, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. Nonlocal terms (such as convolution) model the interactions between the agents. In this article, we focus on this second approach, *i.e.* macroscopic models. To our knowledge, there exist few studies of control of this family of equations. In [38], the authors provide approximate alignment of a crowd described by the macroscopic Cucker-Smale model [22]. The control is the

*Submitted to the editors DATE.

Funding: This work has been carried out in the framework of Archimède Labex (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the "Investissements d'Avenir" French Government programme managed by the French National Research Agency (ANR). The authors acknowledge the support of the ANR project CroCo ANR-16-CE33-0008.

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acceleration, and it is localized in a control region ω which moves in time. In a similar situation, a stabilization strategy has been established in [14, 15], by generalizing the Jurdjevic-Quinn method to partial differential equations. Other forms of control of transport equations with non-local terms have been described in [19, 20] with boundary control. In [17] the authors study optimal control of transport equations with non-local terms in which the control is the non-local term itself.

A different approach is given by mean-field type control, *i.e.* control of mean-field equations and of mean-field games modeling crowds. See *e.g.* [1, 2, 16, 27]. In this case, problems are often of optimization nature, *i.e.* the goal is to find a control minimizing a given cost. In this article, we are mainly interested in controllability problems, for which mean-field type control approaches seem not adapted.

In this article, we study a macroscopic model, thus the crowd is represented by its density, that is a time-evolving measure $\mu(t)$ defined for positive times t on the space $\mathbb{R}^d$ ($d \geqslant 1$). The natural (uncontrolled) velocity field for the measure is denoted by $v : \mathbb{R}^d \to \mathbb{R}^d$, being a vector field assumed Lipschitz and uniformly bounded.

The control acts on the velocity field in a fixed portion ω of the space, which will be a **nonempty open bounded connected subset** of $\mathbb{R}^d$. The admissible controls are thus functions of the form $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ which support in the space variable is included inside ω. We will discuss later the regularity of such control: nevertheless, in the classical approach such control is a Lipschitz function with respect to the space variable in the whole space $\mathbb{R}^d$.

We then consider the following linear transport equation

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where μ^0 is the initial data (initial configuration of the crowd) and the function u is an admissible control. The function $v + \mathbb{1}_\omega u$ represents the velocity field acting on μ. System (1.1) is a first simple approximation for crowd modelling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand controllability properties for such simple equation as a first step, before dealing with velocity fields depending on the crowd itself. Thus, in a future work, we will study controllability of crowd models with a nonlocal term $v[\mu]$, based on the linear results presented here.

Even though System (1.1) is linear, the control acts on the velocity, thus the control problem is nonlinear, which is one of the main difficulties in this study.

The problem presented here has been already studied in very particular cases, when the control acts everywhere. For example, in [35], the author studies the problem of finding a homeomorphism sending a volume form (in our language, a measure that is absolutely continuous with respect to the Lebesgue measure with C^∞ density) to another. In [23], the authors study the same problem on a manifold with boundary, searching for a homeomorphism sending a volume form to another keeping the points on the boundary. Finally, in [9], a parabolic equation is studied: beside the uncontrolled Laplacian term, a transport term is added. The presence of the Laplacian introduces more regularity with respect to our problem, that indeed allows to use solutions of stochastic ODEs instead of classical ones. For this reason, this article is the first characterizing controllability properties of the transport equation with localized controls on the velocity field in presence of an uncontrolled vector field v acting as a drift.

The goal of this work is to study the control properties of System (1.1). We now recall the notion of approximate controllability and exact controllability for System (1.1). We say that System (1.1) is *approximately controllable* from μ^0 to μ^1 on the time interval $[0,T]$ if we can steer the solution to System (1.1) at time T as close to μ^1 as we want with an appropriate control $\mathbb{1}_\omega u$. Similarly, we say that System (1.1) is *exactly controllable* from μ^0 to μ^1 on the time interval $[0,T]$ if we can steer the solution to System (1.1) at time T exactly to μ^1 with an appropriate control $\mathbb{1}_\omega u$. In Definition 2.10 below, we give a formal definition of the notion of approximate controllability in terms of Wasserstein distance.

The main results of this article show that approximate and exact controllability depend on two main aspects: first, from a geometric point of view, the uncontrolled vector field v needs to send the support of μ^0 to ω forward in time and the support of μ^1 to ω backward in time. This idea is formulated in the following condition:

Condition 1.1 (Geometric Condition). Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ satisfying:

(i) For each $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi^v_{t^0}(x^0) \in \omega$, where Φ^v_t is the *flow* associated to v, *i.e.* the solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = v(x(t)) \text{ for a.e. } t > 0, \\ x(0) = x^0. \end{cases}$$

(ii) For each $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi^v_{-t^1}(x^1) \in \omega$.

This geometric aspect is illustrated in Figure 1.

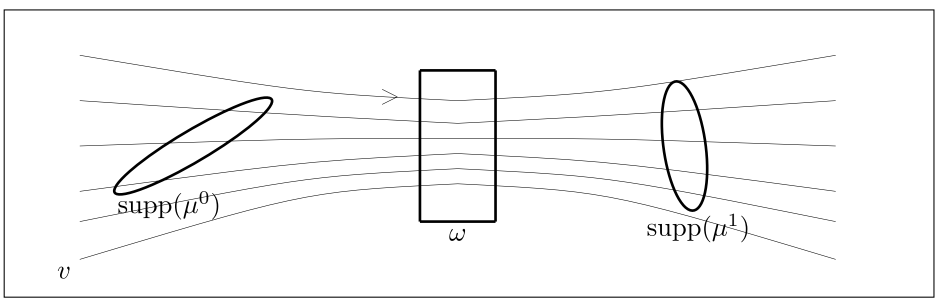

FIG. 1. *Geometric Condition 1.1.*

Remark 1.2. Condition 1.1 is the minimal one that we can expect to steer any initial condition to any target. Indeed, if there exists a point x^0 of the interior of supp(μ^0) for which the first item of the Geometrical Condition 1.1 is not satisfied, then there exists a part of the population of the measure μ^0 that never intersects the control region, thus we cannot act on it.

The second aspect that we want to highlight is the following: The measures μ^0 and μ^1 need to be sufficiently regular with respect to the flow generated by $v + \mathbb{1}_\omega u$. Three cases are particularly relevant:

a) Controllability with Lipschitz controls

If we impose the classical Carathéodory condition of $\mathbb{1}_\omega u$ being Lipschitz in space, measurable in time and uniformly bounded, then the flow $\Phi^{v+\mathbb{1}_\omega u}_t$ is an homeomorphism (see [10, Th. 2.1.1]). As a result, one can expect approximate controllability only, since for general measures there exists no homeomorphism sending one to another. For more details, see Section 4.1. We then have the following result:

THEOREM 1.3 (Main result - Controllability with Lipschitz control). *Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ compactly supported, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.1. Then there exists T such that System (1.1) is **approximately controllable** on the time interval $[0,T]$ from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.*

We give a proof of Theorem 1.3 in Section 3. This proof is a constructive one and strongly uses the fact that the velocity vector field v is autonomous, i.e. not dependent on time. Moreover, it is clear that the extension of our work to time dependent velocity vector fields should require a non-trivial modification of the Geometric Condition 1.1. For the initial measure μ^0 (forward trajectory) the modification is simply the replacement of the flow of the autonomous vector field with the flow of the non-autonomous one, starting from $t = 0$. Instead, for the final measure μ^1 (backward trajectories) one needs to consider the non-autonomous vector field starting from the final time T, which is an unknown of the problem.

Remark 1.4. Due to the finite speed of propagation outside of ω, approximate controllability cannot hold at arbitrary small time. The study of this minimal controllability time is carried on in the forthcoming paper [25].

Remark 1.5. If one removes the assumption of boundedness of v, replacing it with other conditions ensuring boundedness of the flow for each time (*e.g.* by imposing sub-linear growth), then the results presented here still hold. Indeed, it is sufficient to observe that we mainly deal with properties of the flow, that are preserved in this case.

If one instead removes the assumption of boundedness of the supports of μ^0, μ^1 keeping boundedness of v, it is clear that controllability does not hold in general. Indeed, one needs an infinite time to steer the whole mass of μ^0 to the mass of μ^1.

Finally, if one removes both boundedness of the supports and boundedness of the velocity v, it is possible to find examples of approximate controllability in finite time. For example, in $\mathbb{R}^+$ with $\omega = \mathbb{R}^+$, consider the vector field $v(x) = x^2$, for which the flow is $\Phi^v_t(x_0) = \frac{x_0}{1-tx_0}$, defined only for $t < x_0^{-1}$. Thus, one can verify that $\mu^0 = \mathbb{1}_{[0,1]}$ is sent to $\mu^1 = \frac{1}{(x+1)^2}\mathbb{1}_{[0,+\infty)}$ at time $T = 1$. Nevertheless, the problem under such less restrictive hypotheses seems harder to study in its generality, even though adaptations of the method presented here seem possible. Moreover, our applications to crowd modeling and control always assume finite speed of propagation and measures with bounded support.

b) Controllability with vector fields inducing maximal regular flows

To hope to obtain exact controllability of System (1.1) at least for absolutely continuous measures, it is then necessary to search among controls $\mathbb{1}_\omega u$ with less regularity. A weaker condition on the regularity of the velocity field for the well-posedness of System (1.1) has been recently introduced by Ambrosio-Colombo-Figalli in [4], extending previous results by Ambrosio [3] and DiPerna-Lions [24]. Examples of vector fields satisfying such condition are Sobolev vector fields [24], and BV (bounded variation) vector fields with locally integrable divergence [3]. Thus, if we choose the admissible controls satisfying the setting of [4], it is not necessary that there exists an homeomorphism between μ^0 and μ^1.

For all such theories, given a vector field w, a suitable concept of flow Φ_t^w is introduced, such as the maximal regular flow [4], generalizing the regular Lagrangian flow of [3]. Even though such flow does not enjoy all the properties of flows of Lipschitz vector fields, a common requirement is that the Lebesgue measure $\mathcal{L}$ restricted to an open bounded set A is transported to a measure bounded from above by a multiple of the Lebesgue measure itself. In other terms, there exists of a constant $C > 0$ such that for all $t \in [0, T]$ it holds

$$\Phi_t^w \# \mathcal{L}|_A \leqslant C\mathcal{L} \tag{1.2}$$

We will show in Section 4.1 that this condition implies the non-existence of controls exactly steering one absolutely continuous measure to another, for specific choices of μ^0, μ^1. Thus, even this setting does not allow to yield exact controllability.

It is also interesting to observe that Property (1.2) is often required as a necessary condition for a reasonable generalization of the standard theory of Ordinary Differential Equations. Indeed, for Lipschitz vector fields w, the constant C is given by $e^{\mathrm{Lip}(w)t}$. Then, in DiPerna-Lions such condition is required in [24, Eq. (7)] on both sides, while in Ambrosio it is required in [3, Eq (6.1)]. In this sense, the non-exact controllability seems a drawback of a desired condition for an even very general theory of Ordinary Differential Equations, rather than a goal to be reached.

c) Controllability with L^2 controls

We then consider an even larger class of controls, that are general Borel vector fields. In this setting, we have exact controllability under the Geometric Condition 1.1 for any pairs of measures, even not absolutely continuous. Moreover, we prove that one can restrict the set of admissible controls to those that are L^2 with respect to the measure itself, *i.e.* to controls satisfying

$$\int_0^1 \int_{\mathbb{R}^d} |u(t)|^2 d\mu(t) dt < \infty. \tag{1.3}$$

The main drawback is that, in this less regular setting, System (1.1) is not necessarily well-posed. In particular, one has not necessarily uniqueness of the solution. For this reason, one needs to describe solutions to System (1.1) as pairs $(\mathbb{1}_\omega u, \mu)$, where μ is one among the admissible solutions with control $\mathbb{1}_\omega u$.

THEOREM 1.6 (Main result - Controllability with L^2 control). *Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ compactly supported and satisfying Condition 1.1. Then, there exists $T > 0$ such that System* (1.1) *is **exactly controllable** on the time interval $[0, T]$ from μ^0 to μ^1 in the following sense: there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a L^2 vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System* (1.1) *(see Definition 2.6) and satisfying*

$$\mu(T) = \mu^1.$$

A proof of Theorem 1.6 is given in Section 4.

We now resume the main results of the article in the following table.

<table>
<tr><td colspan="2">If μ^0, μ^1 satisfy the Geometric Condition 1.1, then</td></tr>
<tr><td>μ^0, μ^1
absolutely
continuous</td><td>• approx. controllability with Lipschitz control
• NO exact controllability with control inducing maximal regular flows</td></tr>
<tr><td>μ^0, μ^1
general measures</td><td>exact controllability with L^2 control</td></tr>
</table>

This paper is organised as follows. In Section 2, we recall basic properties of the Wasserstein distance and the continuity equation. Section 3 is devoted to the proof of Theorem 1.3, *i.e.* the approximate controllability of System (1.1) with a Lipschitz localized vector field. Finally, in Section 4, we first show that exact controllability does not hold for Lipschitz controls or even vector fields inducing a maximal regular flow; we also prove Theorem 1.6, *i.e.* exact controllability of System (1.1) with a L^2 localized vector field.

2. The Wasserstein distance and the continuity equation. In this section, we recall the definition and some properties of the Wasserstein distance and the continuity equation, which will be used all along this paper. We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in $\mathbb{R}^d$ with compact support and for $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$. We also introduce the classical partial ordering of measures: $\mu \leqslant \nu$ if A being ν-measurable implies A being μ-measurable and $\mu(A) \leqslant \nu(A)$.

We denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} d\pi(x, \cdot) = d\mu(x) \text{ and } \int_{\mathbb{R}^d} d\pi(\cdot, y) = d\nu(y).$$

DEFINITION 2.1. *Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define*

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left\{ \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi \right)^{1/p} \right\}. \tag{2.1}$$

*The quantity is called the **Wasserstein distance**.*

This is the idea of *optimal transportation*, consisting in finding the optimal way to transport mass from a given measure to another. For a thorough introduction, see *e.g.* [41].

We denote by Γ the set of Borel maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$. We now recall the definition of the *push-forward* of a measure:

DEFINITION 2.2. *For a $\gamma \in \Gamma$, we define the push-forward $\gamma\#\mu$ of a measure μ of $\mathbb{R}^d$ as follows:*

$$(\gamma\#\mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ-measurable.

We denote by "AC measures" the measures which are absolutely continuous with respect to the Lebesgue measure and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of AC measures. On $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, the Wasserstein distance can be reformulated as follows:

Property 2.3 (see [41, Chap. 7]). *Let* $p \in [1, \infty)$ *and* $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$*. It holds*

$$W_p(\mu, \nu) = \inf_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}. \tag{2.2}$$

The Wasserstein distance satisfies some useful properties:

Property 2.4 (see [41, Chap. 7]). *Let* $p \in [1, \infty)$*.*

(i) The Wasserstein distance W_p *is a distance on* $\mathcal{P}_c(\mathbb{R}^d)$*.*

(ii) The topology induced by the Wasserstein distance W_p *on* $\mathcal{P}_c(\mathbb{R}^d)$ *coincides with the weak topology.*

(iii) For all $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$*, the infimum in (2.2) is achieved by at least one minimizer.*

The Wasserstein distance can be extended to all pairs of measures μ, ν compactly supported with the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) \neq 0$, by the formula

$$W_p(\mu, \nu) = \mu(\mathbb{R}^d)^{1/p} W_p \left(\frac{\mu}{\mu(\mathbb{R}^d)}, \frac{\nu}{\nu(\mathbb{R}^d)} \right).$$

In the rest of the paper, the following properties of the Wasserstein distance will be also helpful:

Property 2.5 (see [37, 41]). *Let* $\mu,\ \rho,\ \nu,\ \eta$ *be four positive measures compactly supported satisfying* $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ *and* $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$*.*

(i) For each $p \in [1, \infty)$*, it holds*

$$W_p^p(\mu + \rho, \nu + \eta) \leqslant W_p^p(\mu, \nu) + W_p^p(\rho, \eta). \tag{2.3}$$

(ii) For each $p_1,\ p_2 \in [1, \infty)$ *with* $p_1 \leqslant p_2$*, it holds*

$$\begin{cases} W_{p_1}(\mu, \nu) \leqslant W_{p_2}(\mu, \nu), \\ W_{p_2}(\mu, \nu) \leqslant \operatorname{diam}(X)^{1 - p_1/p_2} W_{p_1}^{p_1/p_2}(\mu, \nu), \end{cases} \tag{2.4}$$

where X *contains the supports of* μ *and* ν*.*

We now recall the definition of the continuity equation and the associated notion of weak solutions:

Definition 2.6. *Let* $T > 0$ *and* μ^0 *be a measure in* $\mathbb{R}^d$*. We said that a pair* (μ, w) *composed with a measure* μ *in* $\mathbb{R}^d \times [0, T]$ *and a vector field* $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ *satisfying*

$$\int_0^T \int_{\mathbb{R}^d} |w(t)|\ d\mu(t) dt < \infty$$

*is a **weak solution** to the system, called the **continuity equation**,*

$$\begin{cases} \partial_t \mu + \nabla \cdot (w \mu) = 0 & \text{in } \mathbb{R}^d \times [0, T], \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.5}$$

if for every continuous bounded function $\xi : \mathbb{R}^d \to \mathbb{R}$*, the function* $t \mapsto \int_{\mathbb{R}^d} \xi\ d\mu(t)$ *is absolutely continuous with respect to* t *and for all* $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$*, it holds*

$$\frac{d}{dt} \int_{\mathbb{R}^d} \psi\ d\mu(t) = \int_{\mathbb{R}^d} \langle \nabla \psi, w(t) \rangle\ d\mu(t)$$

for a.e. t *and* $\mu(0) = \mu^0$*.*

Note that $t \mapsto \mu(t)$ is continuous for the weak convergence, it then make sense to impose the initial condition $\mu(0) = \mu^0$ pointwisely in time. Before stating a result of existence and uniqueness of solutions for the continuity equation, we first recall the definition of the flow associated to a vector field.

DEFINITION 2.7. *Let $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ be a vector field being uniformly bounded, Lipschitz in space and measurable in time. We define the* ***flow*** *associated to the vector field w as the application $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that, for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is the solution to the Cauchy problem*

$$\begin{cases} \dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geqslant 0, \\ x(0) = x^0. \end{cases}$$

The following property of the flow will be useful all along the present paper:

PROPERTY 2.8 (see [37]). *Let $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time with a Lipschitz constant equal to L. For each $t \in \mathbb{R}$ and $p \in [1, \infty)$, it holds*

$$W_p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leqslant e^{\frac{(p+1)}{p} L|t|} W_p(\mu, \nu). \tag{2.6}$$

Similarly, let $\mu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and $w_1, w_2 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be two vector fields uniformly bounded, Lipschitz in space with a Lipschitz constant equal to L and measurable in time. Then, for each $t \in \mathbb{R}$ and $p \in [1, +\infty)$, it holds

$$W_p(\Phi_t^{w_1} \# \mu, \Phi_t^{w_2} \# \mu) \leqslant e^{L|t|/p} \frac{e^{L|t|} - 1}{L} \|w_1 - w_2\|_{C^0}. \tag{2.7}$$

We now recall a standard result for the continuity equation:

THEOREM 2.9 (see [41, Th. 5.34]). *Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a vector field uniformly bounded, Lipschitz in space and measurable in time. Then, System (2.5) admits a unique solution μ in $\mathcal{C}^0([0, T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ is equipped with the weak topology. Moreover:*

(i) If $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then the solution μ to (2.5) belongs to $\mathcal{C}^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$.
(ii) We have $\mu(t) = \Phi_t^w \# \mu^0$ for all $t \in [0, T]$.

We now recall the precise notions of approximate controllability and exact controllability for System (1.1):

DEFINITION 2.10. *We say that:*

- *System (1.1) is* **approximately controllable** *from μ^0 to μ^1 on the time interval $[0, T]$ if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ such that the corresponding solutions μ to System (1.1) satisfies*

$$W_p(\mu^1, \mu(T)) \leqslant \varepsilon. \tag{2.8}$$

- *System (1.1) is* **exactly controllable** *from μ^0 to μ^1 on the time interval $[0, T]$ if there exists a control $\mathbb{1}_\omega u$ such that the corresponding solution to System (1.1) is equal to μ^1 at time T.*

It is interesting to remark that, by using properties (2.4) of the Wasserstein distance, estimate (2.8) can be replaced by:

$$W_1(\mu^1, \mu(T)) \leqslant \varepsilon.$$

Thus, in this work, we study approximate controllability by considering the distance W_1 only.

Remark 2.11. One can be interested in proving approximate controllability for a smaller set of controls, for example of class C^k in the space variable with some $k \geqslant 1$. Due to the estimate (2.7), the result of Theorem 1.3 still holds in this case, by density of C^k functions in the space of Lipschitz function with respect to the C^0 norm. Higher regularity in the time variable can be achieved too with the same techniques.

A careful inspection of our proof shows that controls ensuring approximate controllability are not only measurable in time, but they have a finite number of discontinuities in time, that can be smoothened in a small interval of size τ . The introduced error can be arbitrarily small, by using the fact that $\lim_{\tau\to 0} e^{L\tau/p}(e^{L\tau}-1) = 0$.

3. Approximate controllability with a localized Lipschitz control.

In this section, we study approximate controllability of System (1.1) with localized Lipschitz controls. More precisely, in Sections 3.1, we consider the case where the open connected control subset ω contains the support of both μ^0 and μ^1. We then prove Theorem 1.3 in Section 3.2.

3.1. Approximate controllability with a Lipschitz control.

In this section, we prove approximate controllability of System (1.1) with a Lipschitz control, when the open connected control subset ω contains the support of both μ^0 and μ^1. Without loss of generality, we can assume that the vector field v is identically zero by replacing u with $u - v$ in the control set ω.

We then study approximate controllability of system

$$\left\{\begin{array}{ll} \partial_t\mu + \mathrm{div}(u\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d. \end{array}\right. \tag{3.1}$$

PROPOSITION 3.1. *Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ compactly supported in ω. Then, for all $T > 0$, System (3.1) is approximately controllable on the time interval $[0, T]$ from μ^0 to μ^1 with a control $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time. Moreover, the solution μ to System (3.1) satisfies*

$$\mathrm{supp}(\mu(t)) \subset \omega,$$

for all $t \in [0, T]$.

Proof of Proposition 3.1. We assume that $d := 2$, but the reader will see that the proof can be clearly adapted to dimension one or to any other space dimension. In view to simplify the computations, we suppose that $T := 1$ and $\mathrm{supp}(\mu^i) \subset (0, 1)^2 \subset\subset \omega$ for $i = 1, 2$.

We first partition $(0, 1)^2$. Let $n \in \mathbb{N}^*$, consider $a_0 := 0$, $b_0 := 0$ and define the points a_i, b_i for all $i \in \{1, ..., n\}$ by induction as follows: suppose that for a given $i \in \{0, ..., n-1\}$ the points a_i and b_i are defined, then the points a_{i+1} and b_{i+1} are the smallest values such that

$$\int_{(a_i,a_{i+1})\times\mathbb{R}} d\mu^0 = \frac{1}{n} \quad \text{and} \quad \int_{(b_i,b_{i+1})\times\mathbb{R}} d\mu^1 = \frac{1}{n}.$$

Again, for each $i \in \{0, ..., n-1\}$, we consider $a_{i,0} := 0$, $b_{i,0} := 0$ and supposing that for a given $j \in \{0, ..., n-1\}$ the points $a_{i,j}$ and $b_{i,j}$ are already defined, $a_{i,j+1}$ and $b_{i,j+1}$ are the smallest values such that

$$\int_{A_{ij}} d\mu^0 = \frac{1}{n^2} \quad \text{and} \quad \int_{B_{ij}} d\mu^1 = \frac{1}{n^2},$$

where $A_{ij} := (a_i, a_{i+1}) \times (a_{ij}, a_{i(j+1)})$ and $B_{ij} := (b_i, b_{i+1}) \times (b_{ij}, b_{i(j+1)})$. Since μ^0 and μ^1 have a mass equal to 1 and are supported in $(0,1)^2$, then $a_n, b_n \leqslant 1$ and $a_{i,n},\ b_{i,n} \leqslant 1$ for all $i \in \{0, ..., n-1\}$. We give in Figure 2 an example of such partition.

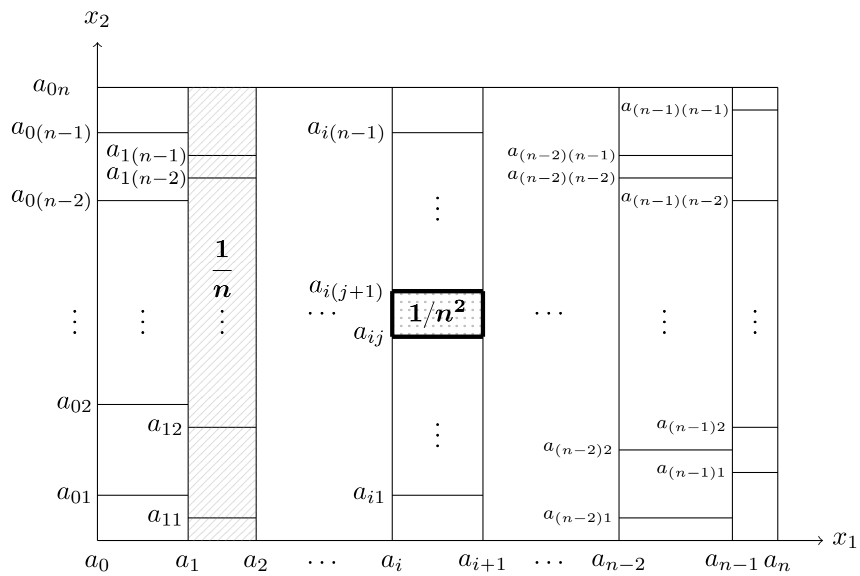

FIG. 2. *Example of a partition for μ^0.*

If one aims to define a vector field sending each A_{ij} to B_{ij}, then some shear stress is naturally introduced, as described in Remark 3.2. To overcome this problem, we first define sets $\widetilde{A}_{ij} \subset\subset A_{ij}$ and $\widetilde{B}_{ij} \subset\subset B_{ij}$ for all $i, j \in \{0, ..., n-1\}$. We then send the mass of μ^0 from each $\widetilde{A}_{ij}$ to $\widetilde{B}_{ij}$, while we do not control the mass contained in $A_{ij} \backslash \widetilde{A}_{ij}$. More precisely, for all $i, j \in \{0, ..., n-1\}$, we define, as in Figure 3, $a_i^-,\ a_i^+, a_{ij}^-,\ a_{ij}^+$ the smallest values such that

$$\int_{(a_i, a_i^-) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \int_{(a_i^+, a_{i+1}) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \frac{1}{n^3}$$

and

$$\int_{(a_i^-, a_i^+) \times (a_{ij}, a_{ij}^-)} d\mu^0 = \int_{(a_i^-, a_i^+) \times (a_{ij}^+, a_{i(j+1)})} d\mu^0 = \frac{1}{n} \times \left(\frac{1}{n^2} - \frac{2}{n^3}\right).$$

We similarly define $b_i^+,\ b_i^-,\ b_{ij}^+,\ b_{ij}^-$ and finally define

$$\widetilde{A}_{ij} := (a_i^-, a_i^+) \times (a_{ij}^-, a_{ij}^+) \text{ and } \widetilde{B}_{ij} := (b_i^-, b_i^+) \times (b_{ij}^-, b_{ij}^+).$$

The goal is to build a solution to System (3.1) such that the corresponding flow Φ_t^u satisfies

$$\Phi_T^u(\widetilde{A}_{ij}) = \widetilde{B}_{ij}, \tag{3.2}$$

for all $i, j \in \{0, ..., n-1\}$. We observe that we do not take into account the displacement of the mass contained in $A_{ij} \backslash \widetilde{A}_{ij}$. We will show that the mass of the corresponding

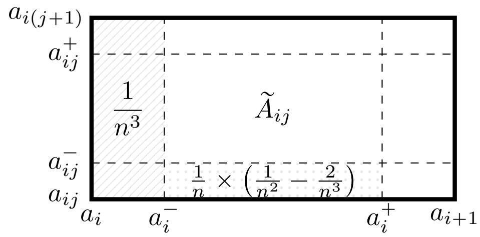

FIG. 3. *Example of cell.*

term tends to zero when n goes to infinity. The rest of the proof is divided into two steps. In a first step, we build a flow satisfying (3.2), then the corresponding vector field. In a second step, we compute the Wasserstein distance between μ^1 and $\mu(T)$, showing that it converges to zero when n goes to infinity. **Step 1:** We first build a flow satisfying (3.2). We recall that $T := 1$. For each $i \in \{0, ..., n-1\}$, we denote by c_i^- and c_i^+ the linear functions equal to a_i^- and a_i^+ at time $t = 0$ and equal to b_i^- and b_i^+ at time $t = T = 1$, respectively, *i.e.* the functions defined for all $t \in [0, T]$ by:

$$c_i^-(t) = (b_i^- - a_i^-)t + a_i^- \quad \text{and} \quad c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+.$$

Similarly, for all $i, j \in \{0, ..., n-1\}$, we denote by c_{ij}^- and c_{ij}^+ the linear functions equal to a_{ij}^- and a_{ij}^+ at time $t = 0$ and equal to b_{ij}^- and b_{ij}^+ at time $t = T = 1$, respectively, *i.e.* the functions defined for all $t \in [0, T]$ by:

$$c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^- \quad \text{and} \quad c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+.$$

Consider the application being the following linear combination of c_i^-, c_i^+ and c_{ij}^-, c_{ij}^+ on $\widetilde{A}_{ij}$, *i.e.*

$$x(x^0, t) := \begin{pmatrix} x_1(x^0, t) \\ x_2(x^0, t) \end{pmatrix} = \begin{pmatrix} \dfrac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \dfrac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t) \\ \dfrac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \dfrac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t) \end{pmatrix}, \tag{3.3}$$

where $x^0 = (x_1^0, x_2^0) \in \widetilde{A}_{ij}$. Let us prove that an extension of the application $(x^0, t) \mapsto x(x^0, t)$ is a flow associated to a vector field u. After some computations, we obtain

$$\begin{cases} \dfrac{dx_1}{dt}(x^0, t) = \alpha_i(t) x_1(x^0, t) + \beta_i(t) & \forall t \in [0, T], \\ \dfrac{dx_2}{dt}(x^0, t) = \alpha_{ij}(t) x_2(x^0, t) + \beta_{ij}(t) & \forall t \in [0, T], \end{cases}$$

where for all $t \in [0, T]$,

$$\begin{cases} \alpha_i(t) = \dfrac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, \quad \beta_i(t) = \dfrac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\ \alpha_{ij}(t) = \dfrac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, \quad \beta_{ij}(t) = \dfrac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}. \end{cases}$$

The last quantities are well defined since for all $i, j \in \{0, ..., n-1\}$ and $t \in [0, T]$

$$\begin{cases} |c_i^+(t) - c_i^-(t)| \geqslant \max\{|a_i^+ - a_i^-|, |b_i^+ - b_i^-|\}, \\ |c_{ij}^+(t) - c_{ij}^-(t)| \geqslant \max\{|a_{ij}^+ - a_{ij}^-|, |b_{ij}^+ - b_{ij}^-|\}. \end{cases}$$

For all $t \in [0, T]$, consider the set

$$\widetilde{C}_{ij}(t) := (c_i^-(t), c_i^+(t)) \times (c_{ij}^-(t), c_{ij}^+(t)).$$

We remark that $\widetilde{C}_{ij}(0) = \widetilde{A}_{ij}$ and $\widetilde{C}_{ij}(T) = \widetilde{B}_{ij}$. On

$$\widetilde{C}_{ij} := \{(x, t) : t \in [0, T], x \in \widetilde{C}_{ij}(t)\},$$

we then define the vector field u by

$$\begin{cases} u_1(x, t) = \alpha_i(t)x_1 + \beta_i(t), \\ u_2(x, t) = \alpha_{ij}(t)x_2 + \beta_{ij}(t), \end{cases}$$

for all $(x, t) \in \widetilde{C}_{ij}$ $(x = (x_1, x_2))$. Notice that the sets $\widetilde{C}_{ij}$ do not intersect. Thus, we extend u by a uniform bounded $\mathcal{C}^\infty$ function outside $\cup_{ij}\widetilde{C}_{ij}$, then u is a $\mathcal{C}^\infty$ function and it satisfies $\mathrm{supp}(u) \subset \omega$.

Then, System (1.1) admits an unique solution and the flow on $\widetilde{C}_{ij}$ is given by (3.3).

Step 2: We now prove that the refinement of the grid provides convergence to the target μ^1, *i.e.*

$$W_1(\mu^1, \mu(T)) \underset{n\to\infty}{\longrightarrow} 0.$$

We remark that

$$\int_{\widetilde{B}_{ij}} d\mu(T) = \int_{\widetilde{B}_{ij}} d\mu^1 = \frac{1}{n^2} - \frac{2}{n^3} - \frac{2}{n}\left(\frac{1}{n^2} - \frac{2}{n^3}\right) = \frac{(n-2)^2}{n^4}.$$

Hence, by defining

$$R := (0, 1)^2 \setminus \bigcup_{ij} \widetilde{B}_{ij},$$

we also have

$$\int_R d\mu(T) = \int_R d\mu^1 = 1 - \frac{(n-2)^2}{n^2}.$$

Using (2.3), it holds

$$W_1(\mu^1, \mu(T)) \quad \leqslant \quad \sum_{i,j=1}^{n} W_1(\mu^1_{|\widetilde{B}_{ij}}, \mu(T)_{|\widetilde{B}_{ij}}) + W_1(\mu^1_{|R}, \mu(T)_{|R}). \tag{3.4}$$

We now estimate each term in the right-hand side of (3.4). Since we deal with AC measures, using Properties 2.4, there exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$, for all $i,j \in \{0,...,n-1\}$, and $\overline{\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{cases} \gamma_{ij}\#(\mu^1_{|\widetilde{B}_{ij}}) = \mu(T)_{|\widetilde{B}_{ij}}, \\ W_1(\mu^1_{|\widetilde{B}_{ij}}, \mu(T)_{|\widetilde{B}_{ij}}) \\ \qquad = \displaystyle\int_{\widetilde{B}_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \end{cases} \quad \text{and} \quad \begin{cases} \overline{\gamma}\#(\mu^1_{|R}) = \mu(T)_{|R}, \\ W_1(\mu^1_{|R}, \mu(T)_{|R}) \\ \qquad = \displaystyle\int_{R} |x - \overline{\gamma}(x)| d\mu^1(x). \end{cases}$$

In the first term in the right hand side of (3.4), observe that γ_{ij} moves masses inside $\widetilde{B}_{ij}$ only. Thus, for all $i,j \in \{0,...,n-1\}$, using the triangle inequality,

$$\begin{aligned} W_1(\mu^1_{|\widetilde{B}_{ij}}, \mu(T)_{|\widetilde{B}_{ij}}) &= \int_{\widetilde{B}_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ \leqslant [(b_i^+ - b_i^-) + (b_{ij}^+ - b_{ij}^-)] \int_{\widetilde{B}_{ij}} d\mu^1(x) &\leqslant (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4}. \end{aligned} \tag{3.5}$$

For the second term in the right-hand side of (3.4), observe that $\overline{\gamma}$ moves a small mass in the bounded set $(0,1)$. Thus it holds

$$W_1(\mu^1_{|R}, \mu(T)_{|R}) = \int_R |x - \overline{\gamma}(x)| d\mu^1(x) \leqslant 2 \left(1 - \frac{(n-2)^2}{n^2}\right) = 8\frac{n-1}{n^2}. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} W_1(\mu^1, \mu(T)) &\leqslant \left(\sum_{i,j=1}^{n} (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4} \right) + 8\frac{n-1}{n^2} \\ &\leqslant 2n\frac{(n-2)^2}{n^4} + 8\frac{n-1}{n^2} \underset{n \to \infty}{\longrightarrow} 0. \end{aligned}$$

$\square$

Remark 3.2. It is not possible in general to build a Lipschitz vector field sending directly each A_{ij} to B_{ij} using the strategy developed in the proof of Proposition 3.1. Indeed, we would obtain discontinuous velocities on the lines c_i. Figure 4 illustrates this phenomenon in the case $n = 2$.

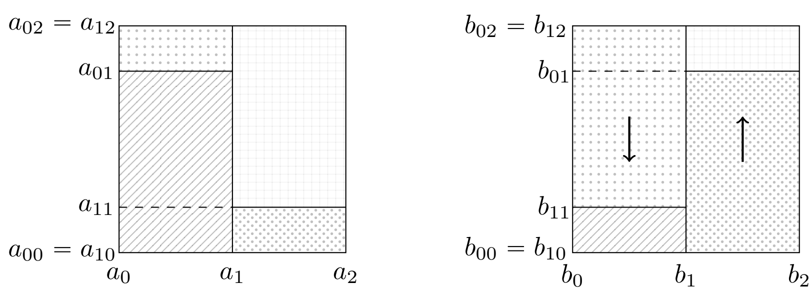

FIG. 4. *Shear stress (left: μ^0, right: μ^1)*

3.2. Approximate controllability with a localized regular control. This section is devoted to prove Theorem 1.3: we aim to prove approximate controllability of System (1.1) with a Lipschitz localized control. This means that we remove the constraints $\text{supp}(\mu^0) \subset \omega$, $\text{supp}(\mu^1) \subset \omega$ and $v := 0$, that we used in Section 3.1. On the other side, we impose Condition 1.1. Before the main proof, we need three useful results. First of all, we give a consequence of Condition 1.1:

Condition 3.3. There exist two real numbers T_0^*, $T_1^* > 0$ and a nonempty open set $\omega_0 \subset\subset \omega$ such that

(i) For each $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 \in [0, T_0^*]$ such that $\Phi_{t^0}^v(x^0) \in \omega_0$, where Φ_t^v is the flow associated to v.

(ii) For each $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 \in [0, T_1^*]$ such that $\Phi_{-t^1}^v(x^1) \in \omega_0$.

LEMMA 3.4. *If Condition 1.1 is satisfied for μ^0, $\mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$, then Condition 3.3 is satisfied too.*

Proof. We use a compactness argument. Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and assume that Condition 1.1 holds. Let $x^0 \in \text{supp}(\mu^0)$. Using Condition 1.1, there exists $t^0(x^0) > 0$ such that $\Phi_{t^0(x^0)}^v(x^0) \in \omega$. Choose $r(x^0) > 0$ such that $B_{r(x^0)}(\Phi_{t^0(x^0)}^v(x^0)) \subset\subset \omega$, where $B_r(x^0)$ denotes the open ball of radius $r > 0$ centered at point x^0 in $\mathbb{R}^d$. Such $r(x^0)$ exists, since ω is open. By continuity of the application $x^1 \mapsto \Phi_{t^0(x^0)}^v(x^1)$ (see [10, Th. 2.1.1]), there exists $\hat{r}(x^0)$ such that

$$x^1 \in B_{\hat{r}(x^0)}(x^0) \quad \Rightarrow \quad \Phi_{t^0(x^0)}^v(x^1) \in B_{r(x^0)}(\Phi_{t^0(x^0)}^v(x^0)).$$

Since μ^0 is compactly supported, we can find a set $\{x_1^0, ..., x_{N_0}^0\} \subset \text{supp}(\mu^0)$ such that

$$\text{supp}(\mu^0) \subset \bigcup_{i=1}^{N_0} B_{\hat{r}(x_i^0)}(x_i^0).$$

We similarly build a set $\{x_1^1, ..., x_{N_1}^1\} \subset \text{supp}(\mu^1)$. Thus Condition 3.3 is satisfied for

$$T_k^* := \max\{t^k(x_i^k) : i \in \{1, ..., N_k\}\},$$

with $k = 0, 1$ and

$$\omega_0 := \left(\bigcup_{i=1}^{N_0} B_{r(x_i^0)}(\Phi_{t^0(x_i^0)}^v(x_i^0)) \right) \bigcup \left(\bigcup_{i=1}^{N_1} B_{r(x_i^1)}(\Phi_{-t^1(x_i^1)}^v(x_i^1)) \right) \subset\subset \omega.$$

∎

The second useful result is the following proposition, showing that we can store a large part of the mass of μ^0 in ω, under Condition 3.3.

PROPOSITION 3.5. *Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the first item of Condition 3.3. Then, for all $\varepsilon > 0$, there exists a space-dependent vector field $\mathbb{1}_\omega u$ Lipschitz and uniformly bounded and a Borel set $A \subset \mathbb{R}^d$ such that*

$$\mu^0(A) = \varepsilon \text{ and } \text{supp}(\Phi_{T_0^*}^{v+\mathbb{1}_\omega u} \# \mu^0_{|A^c}) \subset \omega. \tag{3.7}$$

Proof. For each $k \in \mathbb{N}^*$, we denote by ω_k the closed set defined by

$$\omega_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega_0^c) \geqslant 1/k\}$$

and a cutoff function $\theta_k \in \mathcal{C}^\infty(\mathbb{R}^d)$ satisfying

$$\begin{cases} 0 \leqslant \theta_k \leqslant 1, \\ \theta_k = 1 \text{ in } \omega_0^c, \\ \theta_k = 0 \text{ in } \omega_k. \end{cases}$$

For all $x^0 \in \mathrm{supp}(\mu^0)$, we define

$$t_0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega_0\} \text{ and } t_k(x^0) := \inf\{t \in \overline{\mathbb{R}}^+ : \Phi_t^v(x^0) \in \omega_k\}.$$

For all $k \in \mathbb{N}^*$, we consider

$$u_k := (\theta_k - 1)v \tag{3.8}$$

and

$$S_k := \{x^0 \in \mathrm{supp}(\mu^0)\backslash\omega_0 : \exists s \in (t_0(x^0), t_k(x^0)), \text{ s.t. } \Phi_s^v(x^0) \in \overline{\omega}_0^c\}.$$

The rest of the proof is divided into three steps:

- In Step 1, we prove that the range of the flow associated to x^0 with the control u_k is included in the range of the flow associated to x^0 without control, *i.e.* $\{\Phi_t^{v+u_k}(x^0) : t \geqslant 0\} \subset \{\Phi_t^v(x^0) : t \geqslant 0\}$.
- In Step 2, we show that S_k is a Borel set for all $k \in \mathbb{N}^*$.
- In Step 3, we prove that for a K large enough we have

$$\mu^0(\omega\backslash\omega_K) + \mu^0(S_K) \leqslant \varepsilon. \tag{3.9}$$

Step 1: Consider the flow $y(t) := \Phi_t^v(x^0)$ associated to x^0 without control, *i.e.* the solution to

$$\begin{cases} \dot{y}(t) = v(y(t)),\ t \geqslant 0, \\ y(0) = x^0 \end{cases}$$

and the flow $z_k(t) := \Phi_t^{v+u_k}(x^0)$ associated to x^0 with the control u_k given in (3.8), *i.e.* the solution to

$$\begin{cases} \dot{z}_k(t) = (v + u_k)(z_k(t)) = \theta_k(z_k(t)) \times v(z_k(t)),\ t \geqslant 0, \\ z_k(0) = x^0. \end{cases} \tag{3.10}$$

We use the time change γ_k defined as the solution to the following system

$$\begin{cases} \dot{\gamma_k}(t) = \theta_k(y(\gamma_k(t))),\ t \geqslant 0, \\ \gamma_k(0) = 0. \end{cases} \tag{3.11}$$

Since θ_k and y are Lipschitz, then System (3.11) admits a solution defined for all times. We remark that $\xi_k := y \circ \gamma_k$ is solution to System (3.10). Indeed, for all $t \geqslant 0$ it holds

$$\begin{cases} \dot{\xi}_k(t) = \dot{\gamma}_k(t) \times \dot{y}(\gamma_k(t)) = \theta_k(\xi_k(t)) \times v(\xi_k(t)),\ t \geqslant 0, \\ \xi_k(0) = y(\gamma_k(0)) = y(0). \end{cases}$$

By uniqueness of the solution to System (3.10), we obtain

$$y(\gamma_k(t)) = z_k(t) \text{ for all } t \geqslant 0.$$

Using the fact that $0 \leqslant \theta \leqslant 1$ and the definition of γ_k, we have

$$\begin{cases} \gamma_k \text{ increasing}, & \\ \gamma_k(t) \leqslant t & \forall t \in [0, t_k(x^0)], \\ \gamma_k(t) \leqslant t_k(x^0) & \forall t \geqslant t_k(x^0). \end{cases}$$

We deduce that, for all $x^0 \in \mathrm{supp}(\mu^0)$, it holds

$$\{z_k(t) : t \geqslant 0\} \subset \{y(s) : s \in [0, t_k(x^0)]\}.$$

Step 2: We now prove that S_k is a Borel set by showing that the set

$$R_k := \{x^0 \in \mathbb{R}^d : t_0(x^0) < \infty \text{ and } \exists s \in (t_0(x^0), t_k(x^0)) \text{ s.t. } \Phi_s^v(x^0) \in \overline{\omega}_0^c\}$$

is open. Let $k \in \mathbb{N}^*$, x^0 be an element of R_k and search $r(x^0) > 0$ such that $B_{r(x^0)}(x^0) \subset R_k$.

There exists $s \in (t_0(x^0), t_k(x^0))$ such that $\Phi_s^v(x^0) \in \overline{\omega}_0^c$. Since $\overline{\omega}_0^c$ is open, for a $\beta > 0$, we have $B_\beta(\Phi_s^v(x^0)) \subset \overline{\omega}_0^c$. By continuity of the application $x^1 \mapsto \Phi_s^v(x^1)$, there exists $r(x^0) > 0$ such that

$$x^1 \in B_{r(x^0)}(x^0) \Rightarrow \Phi_s^v(x^1) \in B_\beta(\Phi_s^v(x^0)).$$

Thus, for all $k \in \mathbb{N}^*$, R_k is open. As $S_k = R_k \cap \mathrm{supp}(\mu^0) \cap \omega_0^c$, S_k is a Borel set.

Step 3: We now prove that (3.9) holds for a K large enough. Since we deal with we AC measure, there exists $K_0 \in \mathbb{N}^*$ such that for all $k \geqslant K_0$

$$\mu^0(\omega_0 \backslash \omega_k) \leqslant \varepsilon/2.$$

Argue now by contradiction to prove that there exists $K_1 \geqslant K_0$ such that

$$\mu^0(S_{K_1}) \leqslant \varepsilon/2.$$

Assume that $\mu^0(S_k) > \varepsilon/2$ for all $k \geqslant K_0$. Using the inclusion $S_{k+1} \subset S_k$, we deduce that

$$\mu^0\left(\bigcap_{k \in \mathbb{N}^*} S_k\right) \geqslant \varepsilon/2.$$

Since μ^0 is absolute continuous with respect to λ (the Lebesgue measure), there exists $\alpha > 0$ such that

$$\lambda\left(\bigcap_{k \in \mathbb{N}^*} S_k\right) \geqslant \alpha.$$

We deduce that the intersection of the set S_k is nonempty. Let $\overline{x}^0 \in \mathrm{supp}(\mu^0) \backslash \overline{\omega}_0$ be an element of this intersection. By definition of S_k, for all $k \geqslant K_0$, there exists s_k satisfying

$$\begin{cases} s_k \in (t_0(\overline{x}^0), t_k(\overline{x}^0)), \\ \Phi_{s_k}^v(\overline{x}^0) \in \overline{\omega}_0^c. \end{cases} \tag{3.12}$$

Moreover, the convergence of $t_k(\overline{x}^0)$ to $t_0(\overline{x}^0)$, implies that

$$s_k \to t_0(\overline{x}^0). \tag{3.13}$$

Using the continuity of $x^1 \mapsto \Phi_t^v(x^1)$ and the definition of $t_0(x^0)$, there exists $\beta > 0$ such that

$$\Phi_t^v(\overline{x}^0) \in \omega_0 \text{ for all } t \in (t_0, t_0 + \beta). \tag{3.14}$$

We deduce that (3.14) contradicts (3.12) and (3.13). Thus there exists $K \in \mathbb{N}^*$ such that

$$\mu^0(S_K) + \mu^0(\omega \backslash \omega_K) \leqslant \varepsilon.$$

Since we deal with AC measures, we add a Borel set to have the equality in (3.7), *i.e.* there exists a Borel set S such that

$$\mu^0(S_K \cup \omega \backslash \omega_K \cup S) = \varepsilon.$$

We conclude that, for u defined by

$$u(t) := u^1 := u_K \text{ for all } t \in [0, T_0^*],$$

and $A := S_K \cup \omega \backslash \omega_K \cup S$, Properties (3.7) are satisfied. ∎

The third useful result for the proof of Theorem 1.3 allows to approximately steer a measure contained in ω to a measure contained in an open hypercube $S \subset\subset \omega$.

PROPOSITION 3.6. *Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying $\operatorname{supp}(\mu^0) \subset \omega$. Define an open hypercube S strictly included in $\omega \backslash \operatorname{supp}(\mu^0)$ and choose $\delta > 0$. Then, for all $\varepsilon > 0$, there exists a vector field $\mathbb{1}_\omega u$, Lipschitz and uniformly bounded and a Borel set A such that*

$$\mu^0(A) = \varepsilon \text{ and } \operatorname{supp}(\Phi_\delta^{v+\mathbb{1}_\omega u} \# \mu^0_{|A^c}) \subset S.$$

Proof. Consider S_0 a nonempty open set of $\mathbb{R}^d$ of class $\mathcal{C}^\infty$ strictly included in S and $\widetilde{\omega}$ an open set of $\mathbb{R}^d$ of class $\mathcal{C}^\infty$ satisfying

$$\operatorname{supp}(\mu^0) \cup S \subset\subset \widetilde{\omega} \subset\subset \omega.$$

An example is given in Figure 5. From [28, Lemma 1.1, Chap. 1] (see also [18, Lemma

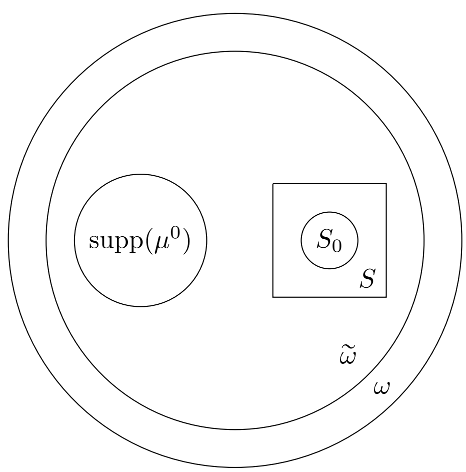

FIG. 5. *Construction of $\widetilde{\omega}$*

2.68, Chap. 2]), there exists a function $\eta \in \mathcal{C}^2(\overline{\widetilde{\omega}})$ satisfying

$$\kappa_0 \leqslant |\nabla \eta| \leqslant \kappa_1 \text{ in } \widetilde{\omega} \backslash S_0, \quad \eta > 0 \text{ in } \widetilde{\omega} \quad \text{and} \quad \eta = 0 \text{ on } \partial \widetilde{\omega}, \tag{3.15}$$

with $\kappa_0, \kappa_1 > 0$. Let $k \in \mathbb{N}^*$. Consider $u_k : \mathbb{R}^d \to \mathbb{R}^d$ Lipschitz and uniformly bounded satisfying

$$u_k := \begin{cases} k\nabla\eta - v & \text{in } \widetilde{\omega}, \\ 0 & \text{in } \omega^c. \end{cases}$$

Let $x^0 \in \mathrm{supp}(\mu^0)$. Consider the flow $z_k(t) = \Phi_t^{v+u_k}(x^0)$ associated to x^0 with the control u_k, *i.e.* the solution to system

$$\text{(3.16)} \qquad \begin{cases} \dot{z}_k(t) = v(z_k(t)) + u_k(z_k(t)), \ t \geqslant 0, \\ z_k(0) = x^0. \end{cases}$$

The different conditions in (3.15) imply that

$$\text{(3.17)} \qquad n \cdot \nabla\eta < C < 0 \text{ on } \partial\widetilde{\omega},$$

where n represents the outward unit normal to $\partial\widetilde{\omega}$. Since $\mathrm{supp}(\mu^0) \subset \widetilde{\omega}$, it holds $z_k(t) \in \widetilde{\omega}$ for all $t \geqslant 0$, otherwise, by taking the scalar product of (3.16) and n on $\partial\widetilde{\omega}$, we obtain a contradiction with (3.17). We now prove that there exists $K(x^0) \in \mathbb{N}^*$ such that for all $k \geqslant K(x^0)$ there exists $t_k(x^0) \in (0, \delta)$ such that $z_k(t_k(x^0))$ belongs to S_0. By contradiction, assume that there exists a sequences $\{k_n\}_{n\in\mathbb{N}^*} \subset \mathbb{N}^*$ such that for all $t \in (0, \delta)$

$$\text{(3.18)} \qquad z_{k_n}(t) \in S_0^c.$$

Consider the function f_n defined for all $t \in [0, \delta]$ by

$$\text{(3.19)} \qquad f_n(t) := k_n \eta(z_{k_n}(t)).$$

Its time derivative is given for all $t \in [0, \delta]$ by

$$\dot{f}_n(t) = k_n \dot{z}_{k_n}(t) \cdot \nabla\eta(z_{k_n}(t)) = k_n^2 |\nabla\eta(z_{k_n}(t))|^2$$

Then, using (3.18), properties (3.15) of η and definition (3.19) of f_n, it holds

$$f_n(\delta) \geqslant k_n^2 \kappa_0^2 \delta \quad \text{and} \quad f_n(\delta) \leqslant k_n \|\eta\|_\infty.$$

We observe that the two last inequalities are in contradiction for n large enough. Then there exists $K(x^0) \in \mathbb{N}^*$ such that for all $k \geqslant K(x^0)$ there exists $t_k(x^0) \in (0, \delta)$ such that $z_k(t_k(x^0))$ belongs to S_0. By continuity, there exists $r(x^0) > 0$ such that $\Phi^{v+u_{K(x^0)}}_{t_{K(x^0)}(x^0)}(x^1)$ belongs to S_0 for all $x^1 \in B_{r(x^0)}(x^0)$. Since $v + u_k$ is linear with respect to k in $\widetilde{\omega}$, then, using the same argument as in Step 1 of the proof of Proposition 3.5, the range of the flow $\Phi^{v+u_k}_\cdot$ is independent of k. Thus, for all $k \geqslant K(x^0)$ there exists $t_k^0(x^0) \in (0, \delta)$ such that $\Phi^{v+u_k}_{t_k^0(x^0)}(x^1) \in S_0$ for all $x^1 \in B_{r(x^0)}(x^0)$. By compactness, there exists $\{x_1^0, ..., x_{N_0}^0\}$ such that

$$\mathrm{supp}(\mu^0) \subset \bigcup_{i=1}^{N_0} B_{r(x_i^0)}(x_i^0).$$

We deduce that for $K := \max_i\{K(x_i^0)\}$, for all $x^0 \in \mathrm{supp}(\mu^0)$ there exists $t^0(x^0)$ for which $\Phi^{v+u_K}_{t^0(x^0)}(x^0)$ belongs to S_0. We remark that the first item of Condition 3.3 holds replacing ω, ω_0 and T_0^* by S, S_0 and δ, respectively. We conclude applying Proposition 3.5 replacing ω, ω_0, T_0^* and v by S, S_0, δ and $v + u_K$, respectively. ∎

Remark 3.7. An alternative method to prove Proposition 3.6 involves building an explicit flow composed with straight lines as in the proof of Proposition 3.1. However, for such method we need to assume that ω is convex, contrarily to the more general approach developed in the proof of Proposition 3.6.

We now have all the tools to prove Theorem 1.3.

Proof of Theorem 1.3. Consider μ^0, μ^1 satisfying Condition 1.1. By Lemma 3.4, there exist T_0^*, T_1^*, ω_0 for which μ^0, μ^1 satisfy Condition 3.3. Let δ, $\varepsilon > 0$ and

$T := T_0^* + T_1^* + \delta$. We now prove that we can construct a Lipschitz uniformly bounded and control $\mathbb{1}_\omega u$ such that the corresponding solution μ to System (1.1) satisfies

$$W_1(\mu(T), \mu^1) \leqslant \varepsilon.$$

Denote by $T_0 := 0$, $T_1 := T_0^*$, $T_2 := T_0^* + \delta/3$, $T_3 := T_0^* + 2\delta/3$, $T_4 := T_0^* + \delta$ and $T_5 := T_0^* + T_1^* + \delta$. Also fix an open hypercube $S \subset\subset \omega \backslash \omega_0$. There exists $R > 0$ such that the supports of μ^0 and μ^1 are strictly included in a hypercube with edges of length R. Define

$$\overline{R} := R + T \times \sup_{\mathbb{R}^d} |v|.$$

Applying Proposition 3.5 on $[T_0, T_1] \cup [T_4, T_5]$ and Proposition 3.6 on $[T_1, T_2] \cup [T_3, T_4]$, we can construct some space-dependent controls u^1, u^2, u^4, u^5 Lipschitz and uniformly bounded, with $\text{supp}(u^i) \subset \omega$, and two Borel sets A_0 and A_1 such that

$$\mu^0(A_0) = \mu^1(A_1) = \frac{\varepsilon}{2d\overline{R}},$$

the solution forward in time to

$$\begin{cases} \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^1)\rho_0) = 0 & \text{in } \mathbb{R}^d \times [T_0, T_1], \\ \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^2)\rho_0) = 0 & \text{in } \mathbb{R}^d \times [T_1, T_2], \\ \rho_0(T_0) = \mu^0_{|A_0^c} & \text{in } \mathbb{R}^d \end{cases}$$

and the solution backward in time to

$$\begin{cases} \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^5)\rho_1) = 0 & \text{in } \mathbb{R}^d \times [T_4, T_5], \\ \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^4)\rho_1) = 0 & \text{in } \mathbb{R}^d \times [T_3, T_4], \\ \rho_1(T_5) = \mu^1_{|A_1^c} & \text{in } \mathbb{R}^d \end{cases}$$

satisfy $\text{supp}(\rho_0(T_2)) \subset S$ and $\text{supp}(\rho_1(T_3)) \subset S$. By conservation of the mass, we remark that $|\rho_0(T_2)| = |\rho_1(T_3)| = 1 - \varepsilon/2d\overline{R}$. We now apply Proposition 3.1 to approximately steer $\rho_0(T_2)$ to $\rho_1(T_3)$ inside S as follows: we find a control u^3 on the time interval $[T_2, T_3]$ satisfying $\text{supp}(u^3) \subset S$ such that the solution ρ to

$$\begin{cases} \partial_t \rho + \nabla \cdot ((v + \mathbb{1}_\omega u^3)\rho) = 0 & \text{in } \mathbb{R}^d \times [T_2, T_3], \\ \rho(T_2) = \rho_0(T_2) & \text{in } \mathbb{R}^d \end{cases}$$

satisfies

$$W_1(\rho(T_3), \rho_1(T_3)) \leqslant \frac{\varepsilon}{2e^{2L(T_5 - T_3)}},$$

where L is the uniform Lipschitz constant for u^4 and u^5. Thus, denoting by u the concatenation of u^1, u^2, u^3, u^4, u^5 on the time interval $[0, T]$, we approximately steer $\mu^0_{|A_0^c}$ to $\mu^1_{|A_1^c}$, since by (2.6) the solution μ to

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u^i)\mu) = 0 & \text{in } \mathbb{R}^d \times [T_{i-1}, T_i], i \in \{1, ..., 5\}, \\ \mu(0) = \mu^0_{|A_0^c} & \text{in } \mathbb{R}^d \end{cases}$$

satisfies

$$W_1(\Phi_T^{v+u} \# \mu^0_{|A_0^c}, \mu^1_{|A_0^c}) = W_1(\mu(T_5), \mu^1_{|A_1^c}) \leqslant e^{2L(T_5 - T_3)} \frac{\varepsilon}{2e^{2L(T_5 - T_3)}} = \frac{\varepsilon}{2}. \tag{3.20}$$

Since we deal with AC measures, using Properties 2.4, there exists a measurable map $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\begin{cases} \gamma\#\mu^1_{|A_1} = \Phi^{v+u}_T\#\mu^0_{|A_0}, \\ W_1(\Phi^{v+u}_T\#\mu^0_{|A_0}, \mu^1_{|A_1}) = \displaystyle\int_{\mathbb{R}^d} |x-\gamma(x)| d\mu^1_{|A_1}(x). \end{cases}$$

We deduce that

$$W_1(\Phi^{v+u}_T\#\mu^0_{|A_0}, \mu^1_{|A_1}) = \int_{\mathbb{R}^d} |x-\gamma(x)| d\mu^1_{|A_1}(x) \leqslant d\overline{R} \times \frac{\varepsilon}{2d\overline{R}} = \frac{\varepsilon}{2}. \tag{3.21}$$

Inequalities (2.3), (3.20) and (3.21) leads to the conclusion:

$$W_1(\Phi^{v+u}_T\#\mu^0, \mu^1) \leqslant W_1(\Phi^{v+u}_T\#\mu^0_{|A_0^c}, \mu^1_{|A_1^c}) + W_1(\Phi^{v+u}_T\#\mu^0_{|A_0}, \mu^1_{|A_1}) \leqslant \varepsilon.$$

$\square$

4. Exact controllability.

In this section, we study exact controllability for System (1.1). In Section 4.1, we show that exact controllability of System (1.1) does not hold for Lipschitz or controls inducing maximal regular flows. In Section 4.2, we prove Theorem 1.6, *i.e.* exact controllability of System (1.1) with a L^2 localized control under some geometric conditions.

4.1. Negative results for exact controllability.

In this section, we show that exact controllability does not hold in general for Lipschitz controls or even vector fields inducing a maximal regular flow. We will see that topological aspects play a crucial role at this level.

a) Non exact controllability with Lipschitz controls

As explained in the introduction, if we impose the classical Carathéodory condition of $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ being uniformly bounded, Lipschitz in space and measurable in time, then the flow $\Phi^{v+\mathbb{1}_\omega u}_t$ is a homeomorphism (see [10, Th. 2.1.1]). More precisely, the flow and its inverse are locally Lipschitz. This implies that the support of μ^0 and $\mu(T)$ are homeomorphic. Thus, if the support of μ^0 and μ^1 are not homeomorphic, then exact controllability does not hold with Lipschitz controls. In particular, we cannot steer a measure which support is connected to a measure which support is composed of two connected components with Lipschitz controls and conversely.

b) Non exact controllability with vector fields inducing maximal regular flows

To hope to obtain exact controllability of System (1.1) at least for AC measures, it is then necessary to search for a control with less regularity. A weaker condition on the regularity of the vector field for the well-posedness of System (1.1) has been given in [4], generalizing previous conditions in [3, 24]. We first briefly recall the main definitions and results of such theory. We then prove that, in such setting, exact controllability between some pairs of AC measures μ^0, μ^1 does not hold, even when the Geometric Condition 1.1 is satisfied.

We first recall the definition of maximal regular field in [4, Def. 4.4], and the corresponding existence result [4, Thm. 5.7]. In our setting, we aim to find a flow that is defined on the whole space $\mathbb{R}^d$ for all times $[0, T]$. Then, we present a simplified version of maximal regular flows, with no hitting time or blow-up of trajectories. The notation is then simplified too.

DEFINITION 4.1. *Let $w : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ be a Borel vector field. We say that a Borel map Φ_t^w is a maximal regular flow relative to w if it satisfies:*

1. *for almost every $x \in \mathbb{R}^d$, the function $\Phi_t^w(x)$ is absolutely continuous with respect to t and it solves the ordinary differential equation $\dot{x} = w(t,x(t))$ with initial condition $\Phi_t^w(x) = x$;*
2. *for any open bounded set $A \subset \mathbb{R}^d$, there exists a compressibility constant $C(A)$ such that for all $t \in [0,T]$, it holds*

$$\Phi_t^w \# \mathcal{L}|_A \leqslant C(A)\mathcal{L}. \tag{4.1}$$

THEOREM 4.2. *Let $w : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ be a Borel vector field satisfying the following conditions:*

a) $\int_0^T \int_A |w(t,x)|\, dx\, dt < \infty$ *for any open bounded set $A \subset \mathbb{R}^d$;*

b) for any non-negative $\bar{\rho} \in L_+^\infty(\mathbb{R}^d)$ with compact support and any closed interval $[a,b] \subset (0,T)$, the continuity equation

$$\partial_t \rho_t + \nabla \cdot (w\rho_t) = 0 \quad \text{in } \mathbb{R}^d \times (a,b)$$

admits at most one weakly continuous solution for $t \in [a,b]$:*

$$t \mapsto \rho_t \in \mathcal{L}^\infty([a,b]; L_+^\infty(\mathbb{R}^d)) \cap \{f \text{ s.t. } \operatorname{supp}(f) \text{ compact subset of } \mathbb{R}^d \times [a,b]\}$$

with $\rho_a = \bar{\rho}$.

c) for any open bounded set $A \subset \mathbb{R}^d$ it holds

$$\operatorname{div}(w(t,.)) \geqslant m(t) \quad \text{in } A, \text{ with } \ L(A) := \int_0^T |m(t)|\, dt < \infty. \tag{4.2}$$

Then, the maximal regular flow Φ_t^w relative to w exists and is unique. Moreover, for any open compact set A, the compressibility constant $C(A)$ in (4.1) can be chosen as $e^{L(A)}$.

For simplicity, we will study two examples of non-controllability in the 1-D setting only. It is then easy to observe that maximal regular flows preserve the order with respect to the initial data, as Lipschitz flows.

PROPOSITION 4.3. *Let w be a Borel vector field satisfying conditions of Theorem 4.2, and Φ_t^w be the associated maximal regular flow. It then holds*

$$x \leqslant y \Rightarrow \Phi_t^w(x) \leqslant \Phi_t^w(y) \quad \text{for almost every pair } x, y \in \mathbb{R}.$$

Proof. Following the proof of [4, Thm. 5.2], build a family of mollified vector fields w_ε for w: they are all Lipschitz, then they preserve the order $x \leqslant y \Rightarrow \Phi_t^{w_\varepsilon}(x) \leqslant \Phi_t^{w_\varepsilon}(y)$ for all $x, y \in \mathbb{R}$, as a classical property of Lipschitz vector fields in $\mathbb{R}$. By letting $w_\varepsilon \rightharpoonup w$ weakly in $L^1((0,T) \times A)$ for all A open bounded, and observing that other hypotheses of the Stability Theorem 6.2 in [4] are satisfied, one has the result. ∎

We are now ready to present two examples of pairs of AC measures μ^0, μ^1 in $\mathbb{R}$ for which exact controllability does not hold with vector fields inducing maximal regular flows.

Example 4.4. For simplicity, we choose $v \equiv 0$ and $\omega = (-2, 2)$ from now on. For the first example, we define $\mu^0 = \mathbb{1}_{[0,1]}\mathcal{L}$ and $\mu^1(x) = \frac{1}{2}x^{-\frac{1}{2}}\mathbb{1}_{(0,1)}\mathcal{L}$. It is clear that the Geometric Condition 1.1 is satisfied. Assume now that a Borel control u

satisfying conditions of Theorem 4.2 steering μ^0 to μ^1 at a given time $T > 0$ exists. Then, the associated maximal regular flow both satisfies $\mu^1 = \Phi_T^u \# \mu^0$ and there exists $C = C((0,1))$ such that $\Phi_T^u \# \mu^0 \leqslant C\mathcal{L}$. Thus, we deduce that $\mu^1 \leqslant C\mathcal{L}$, which is in contradiction with the definition of μ^1.

Example 4.5. It is clear that the previous example is based on the fact that there exists measures that are absolutely continuous with respect to $\mathcal{L}$ and such that their Radon-Nikodym density are L^1 functions that are not L^∞. One can then be interested in proving exact controllability between measures of the form $\rho(x)\mathcal{L}$ with $\rho(x) \in L^\infty(\mathbb{R})$. Also in this case, one has examples of non exact controllability. Indeed, consider again $v \equiv 0$ and $\omega = (-2, 2)$. Define $\nu^0(x) = 2x\mathbb{1}_{[0,1]}\mathcal{L}$ and $\nu^1 = \mathbb{1}_{[0,1]}\mathcal{L}$. We prove now that also in this case, there exists no control inducing maximal regular flows and realizing exact controllability. By contradiction, assume that such control w exists; thus, the associated flow Φ_t^u satisfies $\Phi_T^u \# \nu^0 = \nu^1$. Then

$$\int_0^1 \mathbb{1}_{\{s\,:\,\Phi_T^u(s) \leqslant \Phi_T^u(x)\}} 2s\, ds = \int_0^1 \mathbb{1}_{\{s \leqslant \Phi_T^u(x)\}}\, ds,$$

Recall now that the flow preserves the ordering, then it necessarily holds

$$\int_0^x 2s\, ds = \int_0^{\Phi_T^u(x)} 1\, ds,$$

i.e. $\Phi_T^u(x) = x^2$. If such a flow exists, then one can apply it to μ^0 in the first example. It then holds $\int_0^x 1\, ds = \int_0^{\Phi_T^u(x)} \frac{1}{2} s^{-\frac{1}{2}} ds$, *i.e.* $\Phi_T^u \# \mu^0 = \mu^1$. Thus, Φ_T^u realizes the exact control from μ^0 to μ^1. Contradiction. Then, there exist no control inducing maximal regular flows and exactly steering ν_0 to ν_1.

Example 4.6. One can be interested in finding counterexamples to exact controllability in $\mathbb{R}^d$ with $d > 1$. The Example 4.4 for non exact controllability can be adapted to this setting, by considering $\mu^0 = \mathcal{L}(B_1(0))^{-1}\mathbb{1}_{B_1(0)}\mathcal{L}$ and $\mu^1 = \rho_1(x)\mathcal{L}$ with ρ_1 being a L^1 but not L^∞ function. The counterexample in Example 4.5 can be adapted too, even though computations cannot be carried out easily by applying useful monotony properties.

4.2. Exact controllability with L^2 controls. In this section, we prove Theorem 1.6, *i.e.* exact controllability of System (1.1) in the following sense: there exists a couple $(\mathbb{1}_\omega u, \mu)$ solution to System (1.1) satisfying $\mu(T) = \mu^1$. Before proving Theorem 1.6, we need three useful results. The first one is the following proposition, showing that we can store the whole mass of μ^0 in ω, under Condition 3.3. It is the analogue of Proposition 3.5. In this case, we control the whole mass, but we do not have necessarily uniqueness of the solution to System (1.1).

PROPOSITION 4.7. *Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ satisfying the first item of Condition 3.3. Then there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a L^2 vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System (1.1) and satisfying*

$$\operatorname{supp}(\mu(T_0^*)) \subset \omega.$$

Proof. For each $x^0 \in \mathbb{R}^d$, we denote by

$$\tilde{t}^0(x^0) := \inf\{t \geqslant 0 : \Phi_t^v(x^0) \in \overline{\omega}_0\}$$

and consider the application $\Psi_\cdot(x^0)$ defined for all $t \geqslant 0$ by

$$\Psi_t(x^0) = \begin{cases} \Phi_t^v(x^0) & \text{if } t \leqslant \tilde{t}^0(x^0), \\ \Phi_{\tilde{t}^0(x^0)}^v(x^0) & \text{otherwise.} \end{cases}$$

For all $t \geqslant 0$, the application Ψ_t is a Borel map. Consider μ defined for all $t \geqslant 0$ by

$$\mu(t) := \Psi_t \# \mu^0.$$

We remark that, for all $t, s \in [0, T_0^*]$ such that $t \geqslant s$,

$$\mu(t) = \Psi_{t-s} \# \mu(s). \tag{4.3}$$

Since $\Phi_\cdot^v(x^0)$ is Lipschitz, for all $x^0 \in \mathbb{R}^d$ and $t \in [0, T_0^*]$, it holds

$$|\Psi_t(x^0) - x^0| \leqslant C \min\{t, t^0(x^0)\} \leqslant Ct. \tag{4.4}$$

Combining (4.3) and (4.4), we deduce for all $t, s \in [0, T_0^*]$ with $s \leqslant t$

$$W_2^2(\mu(s), \mu(t)) \leqslant \int_{\mathbb{R}^d} |\Psi_{t-s}(x) - x|^2 \, d\mu(s) \leqslant \sup_{x \in \mathbb{R}^d} |\Psi_{t-s}(x) - x|^2 \leqslant C|t-s|^2.$$

We deduce that the metric derivative $|\mu'|$ of μ defined for all $t \in [0, T_0^*]$ by

$$|\mu'|(t) := \lim_{s \to t} \frac{W_2(\mu(t), \mu(s))}{|t-s|} \tag{4.5}$$

is uniformly bounded on $[0, T_0^*]$. Then μ is an absolute continuous curve on $\mathcal{P}_c(\mathbb{R}^d)$ (see [5, Def. 1.1.1]). Using [5, Th. 8.3.1], there exists a Borel vector $w : \mathbb{R}^d \times (0, T_0^*) \to \mathbb{R}^d$ satisfying

$$\|w(t)\|_{L^2(\mu(t);\mathbb{R}^d)} \leqslant |\mu'|(t) \text{ a.e. } t \in [0, T_0^*]$$

and the couple (w, μ) is a weak solution to

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times [0, T_0^*], \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d. \end{cases} \tag{4.6}$$

By the uniform bound on the metric derivative, it holds that w is a L^2 vector field. Moreover, for all $t \in [0, T_0^*]$, it holds

$$w(t) \in \mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d)) := \overline{\{\nabla\varphi : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu(t);\mathbb{R}^d)}$$

(see [5, Def. 8.4.1]). Consider an open set ω_1 of class $\mathcal{C}^\infty$ satisfying $\omega_0 \subset\subset \omega_1 \subset\subset \omega$. We now prove that $w(t)$ coincides with $v(t)$ in $\mathrm{supp}(\mu(t)) \backslash \overline{\omega}_1$ *a.e.* $t \in [0, T_0^*]$, *i.e.* we can choose $u = 0$ outside ω. Fix $t \in [0, T_0^*]$ and consider $x \in \mathrm{supp}(\mu(t)) \cap \omega_1^c$. There necessarily exists $x^0 \in \mathrm{supp}(\mu^0)$ such that $\Phi_t^v(x^0) = x$, otherwise $x \in \partial\omega_0$. Moreover for a $B := B_r(x^0)$ with $r > 0$ $\Phi_s^v(B) \subset\subset \omega_0^c$ for all $s \in [0, t]$, otherwise there exists $s \in [0, t]$ for which $\Phi_s^v(x^0) \in \partial\omega_0$. Thus

$$\Phi_t^v = \Psi_t \text{ in } B. \tag{4.7}$$

We denote by $A := \Phi_t^v(B)$. We now prove that

$$\Psi_t^{-1}(A) = (\Phi_t^v)^{-1}(A). \tag{4.8}$$

Consider $x \in (\Phi_t^v)^{-1}(A)$. Equality (4.7) implies $\Phi_t^v(x) = \Psi_t(x)$. Then $x \in \Psi_t^{-1}(A)$. Consider now $x \in \Psi_t^{-1}(A)$, which means $\Psi_t(x) \in A$. Using the fact that $A \cap \overline{\omega}_0 \neq 0$, $t < \widetilde{x}^0(x)$. Then $\Psi_t(x) = \Phi_t^v(x)$ and $x \in (\Phi_t^v)^{-1}(A)$. Thus (4.8) holds. By definition of the push forward,

$$\mu_{|A}(t) = \Psi_t \#(\mu^0_{|\Psi_t^{-1}(A)}) \text{ and } (\Phi_t^v \# \mu^0)_{|A} = \Phi_t^v \#(\mu^0_{|\Phi_t^{-1}(A)}).$$

Since $\Psi_t = \Phi_t^v$ on the set $B = (\Phi_t^v)^{-1}(A) = \Psi_t^{-1}(A)$, this implies

$$\mu_{|A}(t) = \Phi_t^v \# \mu^0_{|A}.$$

By compactness of $\mathrm{supp}(\mu(t)) \cap \omega_1^c$, it holds

$$\mu(t)_{|\omega_1^c} = (\Phi_t^v \# \mu^0)_{|\omega_1^c}.$$

We deduce that, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\mathrm{supp}(\varphi) \subset\subset \omega_1^c$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\mu(t) = \int_{\mathbb{R}^d} \langle \nabla \varphi, w\rangle \, d\mu(t) \quad \text{and} \quad \frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\mu(t) = \int_{\mathbb{R}^d} \langle \nabla \varphi, v\rangle \, d\mu(t).$$

If it holds $v \in \mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d))$, then $w(t) = v$, $\mu(t)$ a.e. in $\overline{\omega_1}^c$, and we conclude by taking $u := w - v$ which is supported in ω and is L^2. If now $v \notin \mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d))$, we can write $v = v_1 + v_2$ with $v_1 \in \mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d))$ and $v_2 \in \mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d))^\perp$, where

$$\mathrm{Tan}_{\mu(t)}(\mathcal{P}_c(\mathbb{R}^d))^\perp = \{\nu \in L^2(\mu(t) : \mathbb{R}^d) : \nabla \cdot (\nu \mu(t)) = 0\}$$

(see for instance [5, Prop. 8.4.3]). In other terms, v_2 plays no role in the weak formulation of the continuity equation. Thus, with the same argument, we can prove that $w(t) = v_1$, $\mu(t)$ a.e. in $\overline{\omega_1}^c$ and we conclude by tacking $u := w - v_1$. ∎

The second useful result for the proof of Theorem 1.6 allows to exactly steer a measure contained in ω to a nonempty open convex set $S \subset\subset \omega$. It is the analogue of Proposition 3.6. In this case, as in Proposition 4.7, we control the whole mass, but we do not have necessarily uniqueness of the solution to System (1.1).

PROPOSITION 4.8. *Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ satisfying $\mathrm{supp}(\mu^0) \subset \omega$. Define a nonempty open convex set S strictly included in $\omega \backslash \mathrm{supp}(\mu^0)$ and choose $\delta > 0$. Then there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a L^2 vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System (1.1) satisfying*

$$\mathrm{supp}(\mu(\delta)) \subset S.$$

Proof. Consider S_0 a nonempty open set of $\mathbb{R}^d$ of class $\mathcal{C}^\infty$ strictly included in S and ω_1 an open set of $\mathbb{R}^d$ of class $\mathcal{C}^\infty$ satisfying

$$\mathrm{supp}(\mu^0) \cup S \subset\subset \omega_1 \subset\subset \omega.$$

An example is given in Figure 5. Consider $\eta \in \mathcal{C}^2(\overline{\omega_1})$ defined in the proof of Proposition 3.6 satisfying (3.15). For all $k \in \mathbb{N}^*$, we consider a Lipschitz vector field v_k satisfying

$$v_k := \begin{cases} k\nabla\eta & \text{in } \omega_1, \\ v & \text{in } \omega^c. \end{cases}$$

We denote by

$$\widetilde{t}_k^0(x^0) := \inf\{t \geqslant 0 : \Phi_t^{v_k}(x^0) \in \overline{S}_0\}.$$

For all $x^0 \in \mathbb{R}^d$ and all $k \in \mathbb{N}^*$, consider the application $\Psi_{k,\cdot}(x^0)$ defined for all $t \geqslant 0$ by

$$\Psi_{k,t}(x^0) = \begin{cases} \Phi_t^{v_k}(x^0) & \text{if } t \leqslant \widetilde{t}_k^0(x^0), \\ \Phi_{\widetilde{t}_k^0(x^0)}^{v_k}(x^0) & \text{otherwise.} \end{cases}$$

Using the same argument as in the proof of Proposition 3.6, for K large enough, $\Psi_{K,\delta}(x^0)$ belongs to S for all $x^0 \in \operatorname{supp}(\mu^0)$. Consider μ defined for all $t \in (0, \delta)$ by $\mu(t) := \Psi_{K,t}\#\mu^0$. As in the proof of Proposition 4.7, there exists a vector field u_K such that (u_K, μ) is a weak solution to System (4.6). Moreover $u_K(t) = v_K$, $\mu(t)$ a.e. in $\overline{S}^c$ and a.e. $t \in [0, \delta]$. Thus, we conclude that $(\mathbb{1}_\omega(u_K - v_K), \mu)$ is solution to System (1.1) and $\operatorname{supp}(\mu(\delta)) \subset S$. ∎

The third useful result for the proof of Theorem 1.6 allows to exactly steer a measure contained in a nonempty open convex set $S \subset\subset \omega$ to a given measure contained in S. It is the analogue of Proposition 3.1. In this situation, we obtain exact controllability of System (1.1), but, again, we do not have necessarily uniqueness of the solution to System (1.1).

PROPOSITION 4.9. *Let $\mu^0,\ \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$ satisfying $\operatorname{supp}(\mu^0) \subset S$ and $\operatorname{supp}(\mu^1) \subset S$ for a nonempty open convex set S strictly included in ω. Choose $\delta > 0$. Then there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a L^2 vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System (1.1) and satisfying*

$$\operatorname{supp}(\mu) \subset S \text{ and } \mu(\delta) = \mu^1.$$

Remark 4.10. The proof of Proposition 4.9 can be obtain thanks to the generalized Benamou-Brenier formula (see [8] for the original work and [39, Th. 5.28] for the generalization). For the sake of completeness, we give below a proof of Proposition 4.9 closely related to the proof of [39, Th. 5.28].

Proof of Proposition 4.9. Let π be the optimal plan given in (2.1) associated to the Wasserstein distance between μ^0 and μ^1. For $i \in \{1, 2\}$, we denote by $p_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ the projection operator defined by

$$p_i : (x_1, x_2) \mapsto x_i.$$

Consider the time-evolving measure μ defined for all $t \in [0, \delta]$ by

$$\mu(t) := \frac{1}{\delta}\left[(\delta - t)p_1 + tp_2\right]\#\pi. \tag{4.9}$$

Using [5, Th. 7.2.2], μ is a constant speed geodesic connecting μ^0 and μ^1 in $\mathcal{P}_c(\mathbb{R}^d)$, *i.e.* for all $s, t \in [0, \delta]$

$$W_2(\mu(t), \mu(s)) = \frac{(t - s)}{\delta} W_2(\mu^0, \mu^1).$$

We deduce that the metric derivative $|\mu'|$ of μ (see (4.5)) is uniformly bounded on $[0, \delta]$. Then μ is an absolute continuous curve on $\mathcal{P}_c(\mathbb{R}^d)$ (see [5, Def. 1.1.1]). Thus, using [5, Th. 8.3.1], there exists a Borel vector field $w : \mathbb{R}^d \times (0, \delta) \to \mathbb{R}^d$ such that

$$\|w(t)\|_{L^2(\mu(t);\mathbb{R}^d)} \leqslant |\mu'|(t) \text{ a.e. } t \in [0, \delta]$$

and the couple (w, μ) is a weak solution to

$$\begin{cases} \partial_t\mu + \nabla\cdot(w\mu) = 0 & \text{in } \mathbb{R}^d \times [0, \delta], \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d. \end{cases}$$

By the uniform bound on the metric derivative, it holds that w is an L^2 vector field. Consider $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$0 \leqslant \theta \leqslant 1, \quad \theta = 1 \text{ in } S \quad \text{and} \quad \theta = 0 \text{ in } \omega^c.$$

We remark that μ is supported in S, then the couple $(\mathbb{1}_\omega u, \mu)$ with

$$u := \theta \times (w - v)$$

is solution to

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times [0, \delta], \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d. \end{cases}$$

□

We now have all the tools to prove Theorem 1.6.

Proof of Theorem 1.6. Consider μ^0 and μ^1 satisfying Condition 1.1. Applying Lemma 3.4, Condition 3.3 holds for some ω_0, T_0^* and T_1^*. Let $T := T_0^* + T_1^* + \delta$ with $\delta > 0$ and $T_0,\ T_1,\ T_2,\ T_3,\ T_4,\ T_5$ be the times given in the proof of Theorem 1.3. Using Proposition 4.7 on $[T_0, T_1] \cup [T_4, T_5]$, there exist $\rho_1 \in \mathcal{C}^0([T_0, T_1], \mathcal{P}_c(\mathbb{R}^d))$, $\rho_5 \in \mathcal{C}^0([T_4, T_5], \mathcal{P}_c(\mathbb{R}^d))$ and some space-dependent L^2 controls $u^1,\ u^5$ with

$$\operatorname{supp}(u^1) \cup \operatorname{supp}(u^5) \subset \omega$$

such that $(\mathbb{1}_\omega u^1, \rho_1)$ is a weak solution forward in time to

$$\begin{cases} \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^1)\rho_1) = 0 & \text{in } \mathbb{R}^d \times [T_0, T_1], \\ \rho_1(T_0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases}$$

and $(\mathbb{1}_\omega u^5, \rho_5)$ is a weak solution backward in time to

$$\begin{cases} \partial_t \rho_5 + \nabla \cdot ((v + \mathbb{1}_\omega u^5)\rho_5) = 0 & \text{in } \mathbb{R}^d \times [T_4, T_5], \\ \rho_5(T_5) = \mu^1 & \text{in } \mathbb{R}^d. \end{cases}$$

Moreover $\operatorname{supp}(\rho_1(T_1)) \subset \omega$ and $\operatorname{supp}(\rho_5(T_4)) \subset \omega$. Consider a nonempty open convex set S strictly included in $\omega \backslash \omega_0$. Using Proposition 4.8 on $[T_1, T_2] \cup [T_3, T_4]$, there exist $\rho_2 \in \mathcal{C}^0([T_1, T_2], \mathcal{P}_c(\mathbb{R}^d))$, $\rho_4 \in \mathcal{C}^0([T_3, T_4], \mathcal{P}_c(\mathbb{R}^d))$ and some space-dependent L^2 controls $u^2,\ u^4$ with

$$\operatorname{supp}(u^2) \cup \operatorname{supp}(u^4) \subset \omega$$

such that $(\mathbb{1}_\omega u^2, \rho_2)$ is a weak solution forward in time to

$$\begin{cases} \partial_t \rho_2 + \nabla \cdot ((v + \mathbb{1}_\omega u^2)\rho_2) = 0 & \text{in } \mathbb{R}^d \times [T_1, T_2], \\ \rho_2(T_1) = \rho_1(T_1) & \text{in } \mathbb{R}^d \end{cases}$$

and $(\mathbb{1}_\omega u^4, \rho_4)$ is a weak solution backward in time to

$$\begin{cases} \partial_t \rho_4 + \nabla \cdot ((v + \mathbb{1}_\omega u^4)\rho_4) = 0 & \text{in } \mathbb{R}^d \times [T_3, T_4], \\ \rho_4(T_4) = \rho_5(T_4) & \text{in } \mathbb{R}^d. \end{cases}$$

Moreover $\operatorname{supp}(\rho_2(T_2)) \subset S$ and $\operatorname{supp}(\rho_4(T_3)) \subset S$. Using Proposition 4.9 on $[T_2, T_3]$, there exist $\rho_3 \in \mathcal{C}^0([T_2, T_3], \mathcal{P}_c(\mathbb{R}^d))$ satisfying $\operatorname{supp}(\rho_3) \subset S$ and a L^2 control u^3 with

$$\operatorname{supp}(u^3) \subset \omega$$

such that $(\mathbb{1}_\omega u^3, \rho_3)$ is a weak solution forward in time to

$$\begin{cases} \partial_t \rho_3 + \nabla \cdot ((v + \mathbb{1}_\omega u^3)\rho_3) = 0 & \text{in } \mathbb{R}^d \times [T_2, T_3], \\ \rho_3(T_2) = \rho_2(T_2) & \text{in } \mathbb{R}^d \end{cases}$$

and satisfies $\rho_3(T_3) = \rho_4(T_3)$. Thus the couple $(\mathbb{1}_\omega u, \mu)$ defined by

$$(\mathbb{1}_\omega u, \mu) = (\mathbb{1}_\omega u^i, \rho_i) \text{ in } \mathbb{R}^d \times [T_{i-1}, T_i),\ \ i \in \{1, ..., 5\}$$

is a weak solution to System (1.1) and satisfies $\mu(T) = \mu^1$. ◻

Acknowledgments. The authors thank F. Santambrogio for his interesting comments and suggestions.

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