When does the evoluted set have negligible boundary?*

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***Abstract*— The evoluted set at time T is the union of all images of an initial set A via the flow at times $t \in (0, T)$. Its regularity is for interest for control, being the attainable set in several relevant examples.**

We first show that it is not sufficient for A to have negligible boundary (i.e. zero Lebesgue measure) to ensure that the evoluted set has negligible boundary too. Instead, we prove that such property holds when A is a $C^{1,1}$ domain.

I. Introduction and motivation

The study of the attainable set starting from a point is a crucial problem in control theory, starting from the classical orbit, Rashevsky-Chow and Krener theorems, see [1], [2], [3]. If the initial state is not precisely identified, but lies in a given set, the problem gets even more complicated. The goal of this article is to study such problem in a first, simplified setting.

Here, we consider a fixed vector field $v(x)$ acting on sets via the flow Φ_v^t it generates. Given an initial set A, we aim to describe the evoluted set A^t, that is the set of points reached at times $\tau \in (0, T)$. It was studied e.g. in [4], [5], [6], [7] and in [8, Lemma 1.1]. Its precise definition is given in Definition 2 below.

A first question needs to be answered:

> **(P)**
>
> If the initial set A has a negligible boundary (i.e. its Lebesgue measure is zero), does the evoluted set have a negligible boundary too?

The first, striking result of this article is to provide a negative answer to **(P)**. We show an example of A that has negligible boundary and such that the boundary of the evoluted set is not negligible. Even more surprisingly, the counterexample relies on very low regularity of A, while the vector field v is constant (so extremely regular).

We then turn our attention to find regularity properties of A that ensure that the evoluted set keeps having negligible boundary. We find that it is sufficient for A to be a $C^{1,1}$ domain, which definition is recalled below. Our main result is the following.

Theorem 1: Let $A \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a globally Lipschitz vector field. Then, for every $t > 0$, the boundary of the evoluted set A^t is negligible.

Problem **(P)** is even more interesting when it is interpreted in terms of densities. Assume to have an initial state that is a probability measure on A, e.g. because the initial configuration is not precisely identified, and consider A to be some form of "safety region". If the measure is absolutely continuous with respect to the Lebesgue measure, are we sure that time modulations of the flow do not concentrate along the boundary, then eventually giving a non-zero probability of being close to unsafe configurations? This problem has been addressed in [8], [9], where examples are presented too. This is also one of the main motivations and future applications of the result presented here: develop a theory describing the reachable set starting from a measure under control action. See further results in this direction in [10], [11].

The structure of the article is the following. In Section II, we will precisely define the evoluted set A^t and show via a counterexample that **(P)** is false, in general. In Section III, we will prove the main result of the article, that is Theorem 1. We will draw some conclusions and future directions of research in Section IV.

II. Definition and problem statement

In this article, we consider a fixed autonomous vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ that is globally Lipschitz continuous. It is then standard to define the corresponding flow $\Phi_v^t(x_0)$, that is the function that to each pair (x_0, t) associates the unique solution to the Cauchy problem:

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0. \end{cases}$$

The definition naturally passes to sets: given $A \subset \mathbb{R}^n$, we have

$$\Phi_t^v(A) := \{\Phi_t^v(x_0) \quad \text{s.t.} \quad x_0 \in A\}.$$

The main object studied in this article is defined here.

Definition 2 (Evoluted set): Given a bounded set $A \subset \mathbb{R}^n$ and a time $t > 0$, the *evoluted set* is the set

$$A^t := \bigcup_{\tau \in (0,t)} \Phi_\tau^v(A).$$

A graphical description is given in Figure 1.

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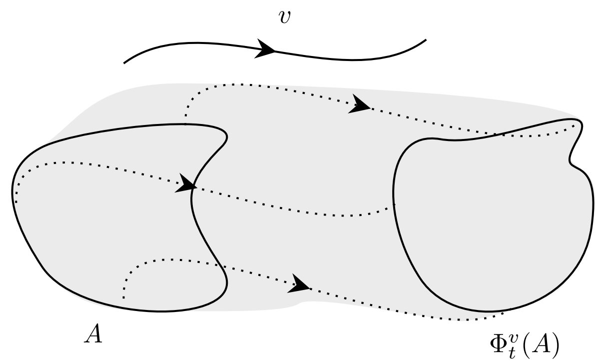

Fig. 1. The evoluted set A^t (in gray).

In terms of control systems, the evoluted set can be either seen as the reachable set from A with times in $(0,t)$, or as the reachable set at time t of the control system:

$$\dot{x}(\tau) = u(\tau)v(x(\tau)), \qquad u(\tau) \in (0,1).$$

See standard results about the geometry of the reachable set from one point in [1], [2], [3].

Remark 3: One can define the evoluted set even for infinite time $t = +\infty$, as

$$A^{+\infty} := \bigcup_{\tau>0} \Phi^v_\tau(A).$$

In this case, Theorem 1 cannot be applied. See a more general result for this case in [7].

Notation: We denote with $B_r(x) = \{z \in \mathbb{R}^n \mid |z-x| < r\}$ the n-dimensional ball of radius r. We denote with $\mathcal{L}_n$ the Lebesgue measure in $\mathbb{R}^n$ and with $\mathscr{H}_{n-1}$ the Hausdorff measure. We also denote with ∂A the boundary of the set A, i.e. $\bar{A} \setminus \mathrm{int}(A)$, that is the set of points in the closure of A not belonging to its interior. Here, the standard topology of $\mathbb{R}^n$ is used, unless a different topology is specified.

A. A counterexample to $(\mathbf{P})$

A first key result of this article is that the statement $(\mathbf{P})$ is false, in general. This is provided by the following counterexample.

We endow $\mathbb{R}^3$ with coordinates (x,y,z), and we denote by $\mathscr{L}_2$ the Lebesgue measure in the (x,y)-plane in what follows. We consider an Osgood curve (see the precise definition in [12]) Γ in the (x,y)-plane given by $\{z=0\}$. For our purpose, it is sufficient to recall that an Osgood curve is a Jordan curve, i.e. the image of an injective continuous map from the circle S^1 to $\mathbb{R}^2$, satisfying $\mathscr{L}_2(\Gamma) > 0$. As a consequence, the interior set A of Γ with respect to the topology of the (x,y)-plane is a well-defined set.

We now consider A and Γ as subsets of $\mathbb{R}^3$. Notice that $A \cup \Gamma$ is closed with respect to the topology of $\mathbb{R}^3$. Clearly, $\partial A = A \cup \Gamma$, and therefore $\mathscr{L}_3(\partial A) = 0$. Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $v(x,y,z) = (0,0,1)$ for every $(x,y,z) \in \mathbb{R}^3$, i.e. $v = \partial_z$. It is clear that the flow of v is the translation $(x,y,z) \to (x,y,z+t)$, hence $A^t = A \times [0,t]$. Therefore for every $t > 0$ the boundary $\partial(A^t)$ contains $(A \cup \Gamma) \times (0,t)$. It then holds

$$\mathscr{L}_3(\partial(A^t)) \geq \mathscr{L}_3(\Gamma \times [0,t]) = \mathscr{L}_2(\Gamma) \cdot t > 0.$$

See a graphical description of the counterexample in Figure 2.

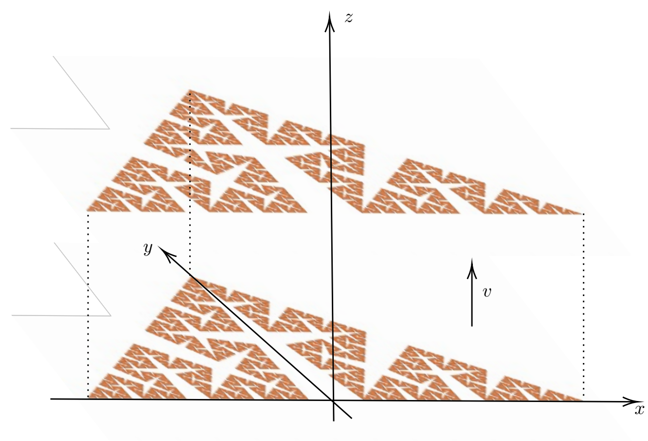

Fig. 2. A counterexample to $(\mathbf{P})$.

This proves that $(\mathbf{P})$ is false, in general. One can then hope to prove $(\mathbf{P})$ under the additional hypothesis that A is open. We now slightly modify the previous example to provide a counterexample to $(\mathbf{P})$ even with A open. Consider the distance function to Γ for points of A, i.e., the function $d : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$d(x,y) := \begin{cases} \inf\limits_{(x_1,y_1)\in\Gamma} |(x,y) - (x_1,y_1)| & (x,y) \in A, \\ 0 & (x,y) \notin A. \end{cases}$$

It is clear that d is Lipschitz, by the triangular inequality.

Define then the open set $B := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in A, -d(x,y) < z < d(x,y)\}$. The boundary ∂B is composed of three pieces, namely $\partial B = S^+ \cup S^- \cup \Gamma$, where

$$S^\pm := \{(x,y,\pm d(x,y)) \mid (x,y,0) \in A\}$$

are the graphs of Lipschitz functions from a two-dimensional set, yielding that $\mathscr{L}_3(S^\pm) = 0$. By using the same vector field $v(x,y,z) = (0,0,1)$, we have that B^t coincides with $B \cup A^t \cup \Phi^v_t(B)$. We then conclude that $\partial(B^t)$ contains $\Gamma \times [0,t]$, and thus again $\mathscr{L}_3(\partial(B^t)) > 0$.

III. Proof of the Main Result

In this section, we prove the main Theorem 1. We find the right regularity condition on A ensuring that $(\mathbf{P})$ is satisfied, by proving that it holds when A is a $C^{1,1}$ domain. We recall the definition here.

Definition 4: A $C^{1,1}$ domain A is an open set which boundary is locally the graph of a C^1 function with Lipschitz derivative.

A. Preliminary results

The proof of the main result relies on the study of some preliminary cases, that are described in this section.

We first prove some topological properties of the evoluted set A^t. We begin stating some basic properties.

Lemma 5: Let $A \subset \mathbb{R}^n$ be a bounded set, and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a globally Lipschitz vector field. Then:

1) If $0 < t_1 < t_2$, then $A^{t_1} \subset A^{t_2}$;
2) For every $B \subset \mathbb{R}^n$, $A \subset B$ implies $A^t \subset B^t$ for every $t > 0$;
3) If A is open, then A^t is open for every $t > 0$;
4) if A is open, then $A \subset A^t$ and $\Phi^v_t(A) \subset A^t$.

Proof: The first two statements are obvious from the definition of the evoluted set. For statement (3), consider the map $h : (0, t) \times A \to (0, t) \times \mathbb{R}^n$ given by $h(\tau, x) := (\tau, \Phi^v_\tau(x))$. Since v is a Lipschitz vector field, then Φ^v_τ is an homeomorphism, hence h is an open map. Recalling that the projection $\pi : (\tau, x) \mapsto x$ is an open map, and noticing that

$$A^t = \pi(h((0, t) \times A)),$$

we conclude that A^t is open as well. For statement (4), let $x \in A$ and $B_\varepsilon(x) \subset A$, that exists since A is open. Consider the trajectory $\Phi^v_\tau(x)$ and observe that there exists $\tau \in (0, t)$ sufficiently small to have $\Phi^v_{-\tau}(x) \in B_\varepsilon(x) \subset A$, hence $x = \Phi^v_\tau(\Phi^v_{-\tau}(x)) \in A^t$. Since x is generic, it holds $A \subset A^t$. The inclusion $\Phi^v_t(A) \subset A^t$ is similar, by observing that $\Phi^v_t(A)$ is open. ∎

We also have the following property of the closure of the evoluted set.

Lemma 6: Let $A \subset \mathbb{R}^n$ be a bounded set, and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz vector field. Then, for every $t > 0$,

$$\overline{A^t} = \bigcup_{\tau \in [0,t]} \overline{\Phi^v_\tau(A)}.$$

As a consequence, $\overline{A^t}$ is compact.

Proof: We begin by proving the $\subset$ inclusion. If $x \in \overline{A^t}$, there exist by definition sequences $(t_n)_{n\in\mathbb{N}} \subset (0, t)$ and $(x_n)_{n\in\mathbb{N}} \subset A$ such that $x = \lim_{n\to\infty} \Phi^v_{t_n}(x_n)$. By compactness of $[0, t]$ and $\overline{A^t}$, also $t_0 := \lim_{n\to\infty} t_n \in [0, t]$ and $x_0 := \lim_{n\to\infty} x_n \in \overline{A}$ exist. We are now going to show that

$$x = \Phi^v_{t_0}(x_0) \in \Phi^v_{t_0}(\overline{A}) = \overline{\Phi^v_{t_0}(A)},$$

completing the claim. Given $\varepsilon > 0$, there exist $n_1(\varepsilon), n_2(\varepsilon), n_3(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \|x - \Phi^v_{t_n}(x_n)\| &\le \varepsilon/3 \quad \text{for every } n \ge n_1(\varepsilon), \quad (1)\\ \|\Phi^v_{t_n}(x_n) - \Phi^v_{t_n}(x_0)\| &\le \varepsilon/3 \quad \text{for every } n \ge n_2(\varepsilon), \quad (2)\\ \|\Phi^v_{t_n}(x_0) - \Phi^v_{t_0}(x_0)\| &\le \varepsilon/3 \quad \text{for every } n \ge n_3(\varepsilon). \quad (3) \end{aligned}$$

Indeed (1) follows by construction, (2) is a consequence of the Gronwall inequality, and (3) is due to che continuity of $t \mapsto \Phi^v_t(x_0)$. By the triangular inequality, we deduce for every $n > n(\varepsilon) := \max\{n_1(\varepsilon), n_2(\varepsilon), n_3(\varepsilon)\}$ that $\|x - \Phi^v_{t_0}(x_0)\| \le \varepsilon$, and we conclude by the arbitrariness of the parameter.

We pass to the $\supset$ inclusion. Let $x \in \bigcup_{\tau\in[0,t]} \overline{\Phi^v_\tau(A)} = \bigcup_{\tau\in[0,t]} \Phi^v_\tau(\overline{A})$. Then there are $y_0 \in \overline{A}$ and $t_0 \in [0, t]$ such that $x = \Phi^v_{t_0}(y_0)$. Since $y_0 \in \overline{A}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ contained in A and converging to y_0. By continuity we deduce that

$$x = \lim_{n\to\infty} \Phi^v_{t_0}(y_n).$$

Now, if $t_0 \in (0, t)$ we are done since all the points $\{\Phi^v_{t_0}(y_n)\}_{n\in\mathbb{N}}$ are in A^t. Then it remains to treat the cases $t_0 \in \{0, t\}$. We only treat the case $t_0 = 0$, the other being analogous: in particular $x = y_0 = \lim_{n\to\infty} y_n$. Define $t_m := t/2m$ for $m \in \mathbb{N}$, and observe that $y_n = \lim_{m\to\infty} \Phi^v_{t_m}(y_n) \in \overline{A^t}$ for every $n \in \mathbb{N}$. Since x can be realised as a limit of points in the closed set $\overline{A^t}$, we conclude that $x \in \overline{A^t}$ as well, as desired. ∎

We end this section with the following crucial result.

Lemma 7: Let $A \subset \mathbb{R}^n$ be an open bounded set, and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a globally Lipschitz vector field. Then, for every $t > 0$, there holds the inclusion

$$\partial(A^t) \subset \partial A \cup \Phi^v_t(\partial A) \cup \left((\partial A)^t \setminus A^t\right). \qquad (4)$$

Proof: Since A^t is open, it holds

$$\partial(A^t) = \overline{A^t} \setminus A^t = \cup_{\tau\in[0,t]} \overline{\Phi^v_\tau(A)} \setminus A^t,$$

where we used Lemma 6. Let $x \in \partial(A^t)$ and $\tau \in [0, t]$ such that $x \in \overline{\Phi^v_\tau(A)}$. By Lemma 5, statement 4, we have $A^t \supset \Phi^v_\tau(A)$ for all $\tau \in [0, t]$. We then write

$$x \in \overline{\Phi^v_\tau(A)} \setminus A^t \subset \Phi^v_\tau(\overline{A}) \setminus \Phi^v_\tau(A) = \Phi^v_\tau(\partial A),$$

where we used that Φ^v_τ is an homeomorphism. This implies

$$\partial(A^t) \subset \partial A \cup \Phi^v_t(\partial A) \cup ((\partial A)^t).$$

Recalling again $\partial(A^t) = \overline{A^t} \setminus A^t$, thus $\partial(A^t) \cap A^t = \emptyset$, we recover (4). ∎

B. The Lebesgue measure of evoluted sets

Thanks to (4), the key point to prove Theorem 1 is to estimate $(\partial A)^t \setminus A^t$. With this goal, in this section we study two cases: vector fields close to zero and vector fields far from zero.

Lemma 8: Let $C \subset \mathbb{R}^n$ be a bounded set of Hausdorff dimension $n - 1$, and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a globally Lipschitz vector field with Lipschitz constant L_v. Then for every $t > 0$, it holds

$$\mathscr{L}_n(C^t) \le t\|v\|_{L^\infty(\overline{C^t})} e^{(n-1)L_v t} \mathscr{H}_{n-1}(C), \qquad (5)$$

where L_v is the Lipschitz constant of v, and $\|v\|_{L^\infty(\overline{C^t})}$ its L^∞ norm over $\overline{C^t}$.

Proof: The statement follows from the estimates

$$\|\Phi^v_t(x) - \Phi^v_t(y)\| \le e^{L_v t}\|x - y\|, \quad \text{and} \qquad (6)$$

$$\|\Phi^v_{\tau_1}(x) - \Phi^v_{\tau_2}(x)\| \le |\tau_1 - \tau_2|\|v\|_\infty, \qquad (7)$$

which are valid for every $t \in (0, +\infty)$, $\tau_1, \tau_2 \in [0, \tau]$, and every pair of points $x, y \in B$. Indeed, equation (6) implies that for every $\tau \in (0, t)$ it holds

$$\mathscr{H}_{n-1}(\Phi^v_\tau(B)) \le e^{(n-1)L_v t} \mathscr{H}_{n-1}(B),$$

while (7) yields, for every $\tau_1, \tau_2 \in (0,t)$, that

$$\begin{aligned} &\operatorname{dist}\left(\Phi^v_{\tau_1}(B), \Phi^v_{\tau_2}(B)\right) \le \\ &\inf\left\{d(\Phi^v_{\tau_1}(x), \Phi^v_{\tau_2}(y)) \mid x, y \in B\right\} \\ \le\quad &\left\{d(\Phi^v_{\tau_1}(x), \Phi^v_{\tau_2}(x)) \mid x \in B\right\} \le t\|v\|_\infty, \end{aligned}$$

and we prove (5). ■

The next lemma is the key technical result that we will use in the sequel. For its proof, we take fully advantage of the $C^{1,1}$ regularity of the domain.

Lemma 9: Let $t > 0$, $A \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and $x \in \partial A$. Let $r > 0$ be such that

$$\mathscr{H}_{n-1}(\partial A \cap \partial B_r(x)) = 0 \tag{8}$$

and $v(y) \neq 0$ for all $y \in \overline{B_r(x)}$. Define $W := A \cap B_r(x)$. It then holds

$$\mathscr{L}_n\left((\partial W)^t \setminus W^t\right) = 0.$$

Proof: Let $t > 0$ be given. Notice that, by assumption (8) and Lemma 8, it holds

$$\mathscr{L}_n((\partial W \cap \partial A \cap \partial B_r(x))^t) \le t \cdot \mathscr{H}_{n-1}(\partial A \cap \partial B_r(x)) = 0.$$

Our result then follows by showing that

$$\mathscr{L}_n\left((\partial W \setminus (\partial A \cap \partial B_r(x)))^t \setminus W^t\right) = 0.$$

For notation's sake we will set $Z := \partial W \setminus (\partial A \cap \partial B_r(x))$ in the rest of the proof. The proof is based on a sequence of inclusions.

Step 1. For every $\varepsilon > 0$, define the sets:

$$Z_\varepsilon := \left\{z \in Z \mid \phi^v_\eta(z) \notin W, \quad \forall\, \eta \in (-\varepsilon, \varepsilon)\right\},$$

and

$$Z_0 := \liminf_{\varepsilon \to 0} Z_\varepsilon = \bigcup_{\varepsilon_0 > 0} \bigcap_{0 < \varepsilon < \varepsilon_0} Z_\varepsilon = \bigcup_{\varepsilon_0 > 0} Z_{\varepsilon_0},$$

where the last equality follows since $Z_{\varepsilon'} \subset Z_\varepsilon$ whenever $\varepsilon' > \varepsilon > 0$. We stress that $Z_0 \subset Z$, and that the following explicit description holds:

$$Z_0 = \left\{z \in Z \mid \exists\ \varepsilon > 0 \text{ s. t. } \ \Phi^v_\eta(z) \notin W, \ \forall\, \eta \in (-\varepsilon, \varepsilon)\right\}.$$

We now claim that

$$Z^t \setminus W^t \subset Z^t_0. \tag{9}$$

Let $y \in Z^t \setminus W^t$, and pick $z \in Z$ and $t_0 \in (0,t)$ such that $y = \Phi^v_{t_0}(z)$. Assume by contradiction that $z \notin Z_0$. Then, we can find a sequence $\{\eta_i\}_{i \in \mathbb{N}}$ of times converging to zero, and such that $\Phi^v_{\eta_i}(z) \in W$ for every $i \in \mathbb{N}$. If we choose η_i so small that $0 < t_0 - \eta_i < t$, we conclude that

$$y = \Phi^v_{t_0}(z) = \Phi^v_{t_0 - \eta_i}(\Phi^v_{\eta_i}(z)) \in W^t.$$

This is a contradiction. Then $z \in Z_0$, hence $y \in Z^t_0$. As y is arbitrary, the inclusion is proved.

Step 2. Consider the function defined on ∂A by

$$f(y) := < v(y), \nu(y) >,$$

where $< .,. >$ is the standard scalar product in $\mathbb{R}^n$ and $\nu(y)$ is the outer normal vector to ∂A in y. Since A is $C^{1,1}$, then $\nu(y)$ exists for all $y \in \partial A$ and is a Lipschitz function. Since v is a Lipschitz function too, the scalar product is Lipschitz.

Remark 10: The regularity of f is the only crucial point of the proof in which we use that A is a $C^{1,1}$ domain.

Define $C := \{x \in W \ \text{ s.t. } \ f(x) = 0\}$. We aim to prove

$$Z^t_0 \subset C^t, \tag{10}$$

by proving that $Z_0 \subset C$. Indeed, if $x \in Z_0 \setminus C$, it satisfies $f(x) \neq 0$. As a consequence, $v(x)$ does not lie in the tangent space $T_x(\partial A)$. By linearization, $\Phi^v_\tau(x) \in W$ for some small (positive or negative) times τ, hence $x \notin Z_0$. Then $Z_0 \setminus C$ is empty and the inclusion is proved.

Step 3. By regularity of f, it holds that C is a Lipschitz domain in ∂A. It then has a boundary ∂C with respect to the topology of ∂A, that is of dimension $n-2$. We denote with $\mathring{C} = C \setminus \partial C$ the interior of C with respect to the topology of ∂A. Define $D := \partial C \cup (\partial C)^t \cup \mathring{C}$. We aim to prove

$$C^t \subset D, \tag{11}$$

i.e. that the evolution of $\mathring{C}$ can be neglected.

Before proving it, we recall an auxiliary result.

Lemma 11: If $y \in \mathring{C}$, then there exists $\varepsilon > 0$ such that $\Phi^v_\tau(y) \in \mathring{C}$ for all $\tau \in (-\varepsilon, \varepsilon)$.

Proof: The proof is classical: restrict v as a vector field on $\mathring{C} \subset \partial A$ only, that is well defined as vectors of v belong to the tangent bundle $T\mathring{C}$. Then find an integral curve starting from y for such restricted vector field. Such curve is also an integral curve of the vector field v defined on the whole $\mathbb{R}^n$, hence $\Phi^v_\tau(y)$ coincides with such curve by uniqueness. By construction, this implies $\Phi^v_\tau(y) \in \mathring{C}$. ■

We are now ready to prove the inclusion (11). If $x \in \partial C$, then $\Phi^v_\tau(x) \in (\partial C)^t \subset D$ for all $\tau \in (0,t)$. If instead $x \in \mathring{C}$, consider the set $\mathcal{T}$ of times $\tau \in [0,t]$ such that $\Phi^v_\tau(x) \in \mathring{C}$. If $\mathcal{T} = [0,t]$, then $\Phi^v_\tau(x) \in \mathring{C} \subset D$ for all $\tau \in (0,t)$. Otherwise, apply Lemma 11 to points of $\Phi^v_\tau(x)$ and observe that $\mathcal{T}$ is strictly contained in $[0,t]$, it is open and it contains 0. Then choose $\varepsilon > 0$ as the infimum of $[0,t] \setminus \mathcal{T}$. This already shows that $\Phi^v_\tau \in \mathring{C} \subset D$ for all $\tau \in [0,\varepsilon)$. By continuity it also holds $f(\Phi^v_\varepsilon(x)) = 0$ but $\Phi^v_\varepsilon(x) \notin \mathring{C}$, hence $\Phi^v_\varepsilon(x) \in \partial C \subset D$. By standard composition of times, it then holds

$$\Phi^v_\tau(x) = \Phi^v_{\tau-\varepsilon}(\Phi^v_\varepsilon(x)) \in (\partial C)^t \subset D$$

for all $\tau \in (\varepsilon, t)$, hence $\Phi^v_\tau(x) \in D$ for all $\tau \in (0,t)$. Since this property holds for any $x \in C$, it holds $C^t \subset D$.

Conclusion. By inclusion (9)-(10)-(11) and by monotonicity of the Lebesgue measure, it holds

$$\mathscr{L}_n\left((\partial W)^t \setminus W^t\right) \le \mathscr{L}_n\left(\partial C \cup (\partial C)^t\right) + \mathscr{L}_n\left(\mathring{C}\right).$$

Since ∂C has Hausdorff dimension $n-2$, it holds $\mathscr{L}_n(\partial C \cup (\partial C)^t) = 0$ by (5). Since it holds $\mathring{C} \subset \partial A$ by definition and ∂A has Hausdorff dimension $n-1$, then its n-Lebsegue measure is zero. ■

C. Proof of Theorem 1

We are now ready to prove our main result. Fix $t > 0$ from now on. Thanks to (4), our goal is to prove $\mathcal{L}_n((\partial A)^t \setminus A^t) = 0$.

Since ∂A is the boundary of a bounded set, it is compact. We now define a covering for it, separated in two parts: the covering of points with nonzero vector field and the one of points with zero vector field.

For each point x satisfying $v(x) \neq 0$, there exists a ball $B_{r(x)}(x)$ such that $v(x) \neq 0$ on the whole closed ball $\overline{B_{r(x)}(x)}$, by continuity. Moreover, eventually slightly reducing the radius, we can always assume that (8) is satisfied: this is a consequence of the coarea formula, see e.g. [13, Thm 2.93]. We denote the set of all these balls by

$$\Omega := \{B_r(x) \ \text{ s.t. } \ v(y) \neq 0 \, \forall y \in \overline{B_{r(x)}(x)} \ \text{ and (8) holds}\}.$$

For points x such that $v(x) = 0$, the construction is a bit more complicated. Define

$$\mathscr{Z} := \{x \in A \ \text{ s.t. } \ v(x) = 0\}.$$

If $\mathscr{Z}$ is nonempty, for a fixed $\varepsilon > 0$ define the open neighborhood of $\mathscr{Z}$ as

$$\mathscr{Z}_\varepsilon := \{y \in \mathbb{R}^n \ \text{ s.t. } \ d(y, \mathscr{Z}) < \varepsilon\}.$$

We now cover ∂A with open sets in $\Omega \cup \{\mathscr{Z}_\varepsilon\}$. Since ∂A is compact, we extract a finite subcovering. It is either of the form $\mathscr{Z}_\varepsilon \cup \bigcup_{i=1}^n B_{r(x_i)}(x_i)$ if $\mathscr{Z}$ is nonempty, or of the form $\bigcup_{i=1}^n (B_{r(x_i)}(x_i)$. Eventually considering $\mathscr{Z}_\varepsilon = \emptyset$, we only study the first case. By intersecting the covering with ∂A itself, we have

$$\partial A \subset (\mathscr{Z}_\varepsilon \cap \partial A) \cup \bigcup_{i=1}^n (B_{r(x_i)}(x_i) \cap \partial A).$$

By following the notation of Lemma 9, we define $W_i := A \cap B_{r(x_i)}(x_i)$. Observe that $B_{r(x_i)}(x_i) \cap \partial A \subset \partial W_i \setminus W_i$, by using the fact that W_i is open. It then holds

$$\begin{aligned} \mathcal{L}_n((\partial A)^t \setminus A^t) \quad \leq \quad & \mathcal{L}_n((\mathscr{Z}_\varepsilon \cap \partial A)^t) + \\ & \sum_{i=1}^n \mathcal{L}_n((\partial W_i)^t \setminus A^t). \end{aligned}$$

Remark that we did not remove A^t from the first term, thus estimating the Lebesgue measure from above. We first observe that

$$\mathcal{L}_n((\partial W_i)^t \setminus A^t) \leq \mathcal{L}_n((\partial W_i)^t \setminus W_i^t) = 0$$

for each $i = 1, \dots, n$. Indeed, it holds $W_i \subset A$, hence $W_i^t \subset A^t$ due to Lemma 5, statement 2. Moreover, hypothesis of Lemma 9 are satisfied, in particular due to the choice of $B_{r(x_i)}(x_i)$ ensuring (8). Then

$$\mathcal{L}_n((\partial A)^t \setminus A^t) \leq \mathcal{L}_n((\mathscr{Z}_\varepsilon \cap \partial A)^t)$$

We are then left to estimate $\mathcal{L}_n((\mathscr{Z}_\varepsilon \cap \partial A)^t)$, for which we aim to apply Lemma 8. With this goal, we need to estimate the L^∞ norm of v on the evoluted set. It is clear that it holds $(\mathscr{Z}_\varepsilon \cap \partial A)^t \subset (\mathscr{Z}_\varepsilon)^t$. Take $\tilde{x} \in (\mathscr{Z}_\varepsilon)^t$ and consider the corresponding $x = \Phi^v_{-\tau}(y) \in \mathscr{Z}_\varepsilon$ with $\tau \in (0, t)$, that exists by definition of the evoluted set. By definition of $\mathscr{Z}_\varepsilon$, there exists $y \in \mathscr{Z}$ such that $\|x - y\| < \varepsilon$. Apply (6), recalling that $\Phi^v_\tau(y) = 0$ due to the fact that it is a zero of the vector field. This implies

$$\|\tilde{x} - y\| \leq e^{L_v \tau} \|x - y\| \leq e^{L_v t} \varepsilon,$$

where L_v is the Lipschitz constant of v. Since $\tilde{x}$ is generic, it holds $(\mathscr{Z}_\varepsilon)^t \subset \mathscr{Z}_{e^{L_v t}\varepsilon}$. A direct computation then implies that

$$\|v\|_{L^\infty((\mathscr{Z}_\varepsilon)^t)} \leq \|v\|_{L^\infty(\mathscr{Z}_{e^{L_v t}\varepsilon})} \leq L_v e^{L_v t} \varepsilon.$$

We are now ready to apply Lemma 8, that gives

$$\begin{aligned} & \mathcal{L}_n((\mathscr{Z}_\varepsilon \cap \partial A)^t) \leq \\ & t \|v\|_{L^\infty((\mathscr{Z}_\varepsilon)^t)} e^{(n-1)L_v t} \mathscr{H}_{n-1}(\mathscr{Z}_\varepsilon \cap \partial A) \\ & \leq t L_v e^{n L_v t} \varepsilon \mathscr{H}_{n-1}(\partial A). \end{aligned}$$

Since $t L_v e^{n L_v t} \mathscr{H}_{n-1}(\partial A)$ is a fixed finite quantity and the estimate holds for any $\varepsilon > 0$, then $\mathcal{L}_n((\mathscr{Z}_\varepsilon \cap \partial A)^t) = 0$ and the result follows.

IV. Conclusions and Future Directions

In this article, we studied the evoluted set and proved that, under suitable hypothesis, its boundary has zero Lebesgue measure. Such result is of particular interest for the use of flows of ordinary differential equations when the initial datum is not reduced to a point but is a probability density.

Future directions of research on this topic will include the case of infinite time horizon and the discussion of weaker regularity properties for the initial datum to ensure that the boundary of the evoluted set is negligible. Applications to control of partial differential equations of transport type, via the method of characteristics, will be also studied.

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