VANISHING VISCOSITY IN MEAN-FIELD OPTIMAL CONTROL

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Abstract. We show the existence of Lipschitz-in-space optimal controls for a class of mean-field control problems with dynamics given by a non-local continuity equation. The proof relies on a vanishing viscosity method: we prove the convergence of the same problem where a diffusion term is added, with a small viscosity parameter.

By using stochastic optimal control, we first show the existence of a sequence of optimal controls for the problem with diffusion. We then build the optimizer of the original problem by letting the viscosity parameter go to zero.

1. Introduction

In recent years, the study of systems describing crowds of interacting agents has drawn a huge interest from the mathematical and control community. A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [4, 6, 17, 28, 34, 35, 46].

Mean-field equations are the natural limit of such systems, composed of a large number N of interacting particles, when N tends to infinity. The state of the system is then a density or, more in general, a measure. Mathematically speaking, the system is often transformed from a large-dimensional ordinary differential equation to a partial differential equation, via the so-called mean-field limit, see e.g. [40, 47].

The finite-dimensional models for interacting agents can either be deterministic (in which the position of each agent is clearly identified), or probabilistic (in which the position of each agent is a probability measure). While deterministic models are based on a (supposedly) perfect knowledge of the dynamics, probabilistic models naturally arise when either individual dynamics or interactions are subjected to some form of noise. As a consequence, mean-field equations have deeply different natures in the two cases: the limit of deterministic models is often a continuity equation, while for probabilistic models it is a diffusion equation. See [21, 22, 40] for a comprehensive introduction.

Beside the dynamics of mean-field equations, it is now relevant to study control problems for them, that are known as *mean-field control problems*. In the mean-field limit for deterministic models, a few articles have been dealing with controllability results [29, 30] or explicit syntheses of control laws [18, 44]. Most of the literature focused on optimal control problems, with contributions ranging from existence results [15, 31, 32, 33] to first-order optimality conditions [7, 11, 12, 13, 14, 23, 24, 45], to numerical methods [1, 16]. The linear-quadratic case is studied in [27] for the deterministic setting and in [7, 8] for the probabilistic one.

Our aim in this article is to develop one more technique to solve mean-field optimal control problems. Indeed, it is natural to expect that, in finite dimension, (uncontrolled) probabilistic models converge (in some sense) to deterministic ones as the noise decreases to zero. See e.g. [41]. The same holds for (uncontrolled) partial differential equations, when solutions to advection-diffusion equations converge to solutions of the continuity equation as the noise parameter (also known as viscosity) goes to zero. This is the basic idea of the *vanishing viscosity method*, see [9, 36]. Observe that, in general, for the partial differential equation with diffusion term, stronger regularity of solution is ensured and better computational methods are available. It is then very

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desirable to be able to use methods for diffusive equations and then pass to the continuity equation by a vanishing viscosity method.

Our aim is exactly to provide a vanishing viscosity result for mean-field control problems. In our article, we deal with two optimal control problems, corresponding to the deterministic and probabilistic settings. On one side, the deterministic optimal control problem is

Problem ($\mathcal{P}$)

Find

$$\min_{u\in\mathcal{A}} J(\mu, u),$$

where the cost J is

$$J(\mu, u) := \int_0^T \int_{\mathbb{R}^d} (f(t, x, \mu_t) + \psi(u(t, x)))\, \mu_t(\,\mathrm{d}x)\,\mathrm{d}t + \int_{\mathbb{R}^d} g(x, \mu_T)\mu_T(\,\mathrm{d}x), \quad (1.1)$$

and $\mu \in C([0, T]; \mathscr{P}_2(\mathbb{R}^d))$ is a solution of

$$\begin{cases} \partial_t \mu_t + \operatorname{div}[(b(t, x, \mu_t) + u(t, x))\mu_t] = 0, \\ \mu|_{t=0} = \mu_0 \end{cases} \quad (1.2)$$

with initial state $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ with compact support.
The set of admissible controls is

$$\mathcal{A} := L^\infty((0, T); L^1(\mathbb{R}^d, U;\, \mathrm{d}\mu_t)),$$

where $U \subset \mathbb{R}^d$.

We now add a viscosity term on the right hand side of (1.2), with $\varepsilon > 0$ being the diffusion parameter, connected to the viscosity of the system. Then, we consider the following problem:

Problem ($\mathcal{P}_\varepsilon$)

Take ($\mathcal{P}$) and replace μ solution of (1.2) with μ^ε solution of

$$\begin{cases} \partial_t \mu_t^\varepsilon + \operatorname{div}[(b(t, x, \mu_t^\varepsilon) + u(t, x))\mu_t^\varepsilon] = \varepsilon\Delta\mu_t^\varepsilon, \\ \mu^\varepsilon|_{t=0} = \mu_0. \end{cases} \quad (1.3)$$

Replace the set of admissible controls $\mathcal{A}$ with

$$\mathcal{A}^\varepsilon := L^\infty((0, T); L^1(\mathbb{R}^d, U;\, \mathrm{d}\mu_t^\varepsilon)).$$

Under natural hypotheses, both solutions (μ, u) of the deterministic problem ($\mathcal{P}$) and solutions $(\mu^\varepsilon, u^\varepsilon)$ of the probabilistic problem ($\mathcal{P}_\varepsilon$) exist. In this framework, the natural questions about vanishing viscosity are the following:

- Do we have convergence of optimal controls $u^\varepsilon \to u$?
- Do we have convergence of optimal trajectories $\mu^\varepsilon \to \mu$?
- Do we have convergence of costs $J(\mu^\varepsilon, u^\varepsilon) \to J(\mu, u)$?

Such questions do not have a general answer. Our main result states that, under quite natural hypotheses, all answers are positive.

Theorem 1.1. *Assume the following:*

- *the set of admissible control values $U \subset \mathbb{R}^d$ is convex and compact;*
- *the vector field b is $C^{1,1}_{\mathrm{loc}}$ regular, i.e.* **Assumption (B)** *in Section 2.3 below holds;*
- *the functions f, ψ, g in J are $C^{1,1}_{\mathrm{loc}}$ regular, i.e.* **Assumption (J)** *in Section 3.1 below holds;*
- *the function ψ is λ-convex, for some $\lambda > 0$, and the functions f, g are convex, i.e.* **Assumption (C)** *in Section 3.1 below holds.*

Let $\Lambda(T,L) := TL(1+2L)e^{(6L+1)T}$, where L is the Lipschitz constant in **Assumption (B), (J)** *below. Then, if $\lambda > \Lambda(T,L)$ there exist:*

- *a unique solution $(\mu^\varepsilon, u^\varepsilon) \in C([0,T];\mathscr{P}_2(\mathbb{R}^d)) \times L^\infty((0,T);\mathrm{Lip}(\mathbb{R}^d,U))$ of $(\mathcal{P}_\varepsilon)$, for each $\varepsilon > 0$;*
- *a solution $(\mu, u) \in C([0,T];\mathscr{P}_2(\mathbb{R}^d)) \times L^\infty((0,T);\mathrm{Lip}(\mathbb{R}^d,U))$ of $(\mathcal{P})$;*

and the following convergences hold:

(i) $u^\varepsilon \rightharpoonup u$ *in* $L^2((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d,U))$ *for every* $1 \le p < \infty$;
(ii) $\mu^\varepsilon \to \mu$ *in* $C([0,T],\mathscr{P}_2(\mathbb{R}^d))$;
(iii) $J(\mu,u) \le \liminf_{\varepsilon\to 0} J(\mu^\varepsilon,u^\varepsilon)$.

It is important to note that the hypothesis on the support of μ_0 can be relaxed (with the necessary technicalities) but we have decided to keep it to make the presentation easier. Moreover, beside standard regularity hypotheses, the most interesting and crucial requirement is certainly strict convexity of the control cost ψ. Indeed, in Section 5 we will also show an example in which the vanishing viscosity limit does not hold, since the cost is convex only. Moreover, the convexity hypothesis on f and g can be relaxed at the price of a larger value for $\Lambda(T,L)$. We will provide further comments in Remark 3.9 below.

The structure of the article is the following. In Section 2 we introduce some standard tools from analysis in the space of probability measures. Moreover, we recall existence and uniqueness results for the non-local continuity (1.2) and advection-diffusion (1.3) equations. In Section 3 we define and solve a class of mean-field stochastic optimal control problems, which is closely related to our original problems $(\mathcal{P})$ and $(\mathcal{P}_\varepsilon)$. Indeed, the results of Section 3 will provide the main building blocks for the proof of our main theorem, which will be given in Section 4. Finally, in Section 5 we show that convergence of optimal controls from $(\mathcal{P}_\varepsilon)$ to $(\mathcal{P})$ is not guaranteed if we drop the strict convexity assumption on the control cost ψ.

2. Notations and preliminaries

In this section, we fix notations and recall some notions of analysis in the space of probability measures, Wasserstein spaces, and non-local continuity equations.

2.1. **The Wasserstein distance.** We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of measures on $\mathbb{R}^d$ and by $\mathscr{P}(\mathbb{R}^d)$ the subset of probability measures. The set of probability measures with compact support is denoted by $\mathscr{P}_c(\mathbb{R}^d)$, while $\mathscr{P}^{\mathrm{ac}}(\mathbb{R}^d)$ denotes the set of probability measures which are absolutely continuous with respect to the d-dimensional Lebesgue measure $\mathscr{L}^d$. We also define $\mathscr{P}^{\mathrm{ac}}_c(\mathbb{R}^d) := \mathscr{P}^{\mathrm{ac}} \cap \mathscr{P}_c(\mathbb{R}^d)$.

We say that a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{P}(\mathbb{R}^d)$ converges in the sense of measures towards $\mu \in \mathscr{P}(\mathbb{R}^d)$, denoted by $\mu^n \overset{*}{\rightharpoonup} \mu$, provided that

$$\lim_n \int_{\mathbb{R}^d} \phi(x)\mu^n(\mathrm{d}x) = \int_{\mathbb{R}^d} \phi(x)\mu(\mathrm{d}x), \quad \text{for all } \phi \in C^\infty_c(\mathbb{R}^d). \tag{2.1}$$

The space $\mathscr{P}(\mathbb{R}^d)$ is equipped with the topology of the convergence of measures. For a given $p \ge 1$, we denote by $\mathscr{P}_p(\mathbb{R}^d)$ the set of probability measures with finite p-th moment M_p, which is defined as

$$M_p(\mu) := \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x). \tag{2.2}$$

Definition 2.1. Let $\mu,\nu \in \mathscr{P}_p(\mathbb{R}^d)$. We say that $\gamma \in \mathscr{P}(\mathbb{R}^{2d})$ is a transport plan between μ and ν provided that $\gamma(A\times\mathbb{R}^d) = \mu(A)$ and $\gamma(\mathbb{R}^d \times B) = \nu(B)$ for any pair of Borel sets $A, B \subset \mathbb{R}^d$. We denote with $\Pi(\mu,\nu)$ the set of such transference plans.

With these notations, we now introduce the Wasserstein distance in the space $\mathscr{P}_p(\mathbb{R}^d)$.

Definition 2.2. Given $p \ge 1$ and two measures $\mu,\nu \in \mathscr{P}_p(\mathbb{R}^d)$, the p-Wasserstein distance between μ and ν is

$$W_p(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \,\gamma(\mathrm{d}x,\mathrm{d}y) \right\}^{1/p}. \tag{2.3}$$

We recall that the Wasserstein distance metrizes the weak-$*$ topology of probability measures; in particular the following holds, see [49, 50].

Proposition 2.3. *The Wasserstein space $(\mathscr{P}_p(\mathbb{R}^d), W_p)$ is a complete and separable metric space. Moreover, for a given $\mu \in \mathscr{P}_p(\mathbb{R}^d)$ and a sequence of measures in $\mu^n \in \mathscr{P}_p(\mathbb{R}^d)$, the following conditions are equivalent:*

- $W_p(\mu, \mu^n) \to 0$, *as* $n \to \infty$,
- $\mu^n \overset{*}{\rightharpoonup} \mu$ *and* $\displaystyle\int_{\mathbb{R}^d} |x|^p \mu^n(\,\mathrm{d}x) \to \int_{\mathbb{R}^d} |x|^p \mu(\,\mathrm{d}x)$,
- $\mu^n \overset{*}{\rightharpoonup} \mu$ *and* $\displaystyle\int_{B_R^c} |x|^p \mu^n(\,\mathrm{d}x) \to 0$ *as* $R \to \infty$ *uniformly in* n.

We recall that Wasserstein distances are *ordered*, in the sense that, given $\mu, \nu \in \mathscr{P}_c(\mathbb{R}^d)$, then

$$W_{p_1}(\mu, \nu) \le W_{p_2}(\mu, \nu), \qquad \text{whenever} \qquad p_1 \le p_2. \tag{2.4}$$

We also denote with $\mathrm{Lip}(\phi)$ a Lipschitz constant for a function ϕ and with $\mathrm{Lip}(X, Y)$ the space of Lipschitz functions from X to Y, as well as $\mathrm{Lip}(X) := \mathrm{Lip}(X, \mathbb{R})$. We now recall the *Kantorovich-Rubinstein* duality formula which characterizes the distance W_1, see [50].

Lemma 2.4. *Let $\mu, \nu \in \mathscr{P}_1(\mathbb{R}^d)$. Then*

$$W_1(\mu, \nu) = \sup_{\phi \in \mathrm{Lip}(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} \phi(x)(\mu - \nu)(\,\mathrm{d}x) : \ \mathrm{Lip}(\phi) \le 1 \right\}. \tag{2.5}$$

2.2. **The L-derivative.** We now recall some results of differential calculus in the space of probability measures. Unless otherwise specified, all definitions and the results are taken from [21]. In particular, we choose a notion of derivative of a functional with respect to a measure, that suits our purposes. We recall that there are several different definitions of derivatives with respect to measures, see e.g. [21]. For our purpose, we need the so-called *L-derivative*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, where atomless means that for any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exists $B \in \mathcal{F}$ such that $0 < \mathbb{P}(B) < \mathbb{P}(A)$.

Definition 2.5. Let $X : \Omega \to \mathbb{R}^d$ be a random variable. We define the *law* of X the measure defined as $\mathcal{L}(X)(B) := \mathbb{P}(X^{-1}(B))$, for any Borel set $B \subset \mathbb{R}^d$.

The following proposition holds, see [10, Proposition 9.1.11].

Proposition 2.6. *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, then there exists a $\mathbb{R}^d$-valued random variable $X \in L^2(\Omega; \mathbb{R}^d)$ with law $\mathcal{L}(X) = \mu$. Moreover, if $\mu, \mu' \in \mathscr{P}_2(\mathbb{R}^d)$, then*

$$W_2(\mu, \mu')^2 = \inf_{(X, X')} \mathbb{E}\left[|X - X'|^2\right],$$

where the infimum is taken over the pairs of $\mathbb{R}^d$-random variables (X, X') such that $\mathcal{L}(X) = \mu$ and $\mathcal{L}(X') = \mu'$.

Given a map $h : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ we define the *lift* $\tilde{h} : L^2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ in the following way

$$\tilde{h}(X) = h(\mathcal{L}(X)), \quad \forall X \in L^2(\Omega; \mathbb{R}^d).$$

Note that $\mathcal{L}(X) \in \mathscr{P}_2(\mathbb{R}^d)$, since $X \in L^2(\Omega; \mathbb{R}^d)$. We point out that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an Hilbert space, in which the notion of Fréchet differentiability makes sense. We thus have the following definition.

Definition 2.7. A function $h : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be *L-differentiable* at $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ if there exists a random variable X_0 with law μ_0 such that the lifted function $\tilde{h}$ is Fréchet differentiable at X_0.

The Fréchet derivative of $\tilde{h}$ at X can be viewed as an element of $L^2(\Omega; \mathbb{R}^d)$; we denote it by $D\tilde{h}(X)$. It is important to recall that L-differentiability of h does not depend upon the particular choice of X, as explained in the following propositions, see [21].

Proposition 2.8. *Let $h : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\tilde{h}$ its lift. Let $X, X' \in L^2(\Omega; \mathbb{R}^d)$ with the same law. If $\tilde{h}$ is Fréchet differentiable at X, then $\tilde{h}$ is Fréchet differentiable at X' and $(X, D\tilde{h}(X))$ has the same law as $(X', D\tilde{h}(X'))$.*

Proposition 2.9. *Let $h : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ be an L-differentiable function. Then, for any $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a measurable function $\xi : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $X \in L^2(\Omega; \mathbb{R}^d)$ with law μ_0, it holds that $D\tilde{h}(X) = \xi(X)$ μ_0-almost surely.*

We say that h is continuously L-differentiable if $D\tilde{h}$ is a continuous function from the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ into itself. Moreover, by Proposition 2.9, the equivalence class of $\xi \in L^2(\mathbb{R}^d, \mu_0; \mathbb{R}^d)$ is uniquely defined; we denote it by $\partial_\mu h(\mu_0)$. We call *L-derivative* of h at μ_0 the function

$$\partial_\mu h(\mu_0)(\cdot) : x \in \mathbb{R}^d \mapsto \partial_\mu h(\mu_0)(x).$$

From the above construction, it is clear that $\partial_\mu h(\mu_0)(\cdot)$ is uniquely defined only μ_0-a.e.. However, if $D\tilde{h}$ is a Lipschitz function from $L^2(\Omega, \mathcal{F}, \mathbb{P})$ into itself, we can define a Lipschitz continuous version of $\partial_\mu h(\mu_0)(\cdot)$. This is the content of the following proposition, see [21].

Proposition 2.10. *Assume that $(v(\mu)(\cdot))_{\mu \in \mathscr{P}_2(\mathbb{R}^d)}$ is a family of Borel-measurable mappings from $\mathbb{R}^d$ into itself for which there exists a constant $C > 0$ such that, for any pair of identically distributed square integrable random variables $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ over an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, it holds:*

$$\mathbb{E}\left[|v(\mathcal{L}(\xi_1))(\xi_1)) - v(\mathcal{L}(\xi_2))(\xi_2)|^2\right] \le C^2 \mathbb{E}\left[|\xi_1 - \xi_2|^2\right].$$

Then, for each $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, one can redefine $v(\mu)(\cdot)$ on a μ-negligible set in such a way that:

$$\forall x, x' \in \mathbb{R}^d, \qquad \text{it holds} \qquad |v(\mu)(x) - v(\mu)(x')| \le C|x - x'|, \tag{2.6}$$

for the same C as above.

To the above definition of differentiability we associate the following definition of convexity.

Definition 2.11. We say that a function $h : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is *L-convex* if it is L-differentiable and satisfies

$$h(\mu') \ge h(\mu) + \mathbb{E}[\partial_\mu h(\mu)(X) \cdot (X' - X)],$$

whenever $X, X' \in L^2(\Omega; \mathbb{R}^d)$ have law μ, μ', respectively.

Finally, it is natural to extend the above definitions to functions depending on an d-dimensional variable x and on a probability measure μ, i.e. of the type $h : (x, \mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. With these notations, a function h is *jointly differentiable* if its lift $\tilde{h} : \mathbb{R}^d \times L^2(\Omega; \mathbb{R}^d)$ is jointly differentiable. In particular, we can define partial derivatives $\partial_x h(x, \mu)$ and $\partial_\mu h(x, \mu)(x')$. We remark that joint continuous differentiability in the two arguments is equivalent to partial differentiability in each of the two arguments and joint continuity of the partial derivatives. Thus, the definitions and the results of this section can be easily generalized to this setting. In particular, if the derivatives of h are Lipschitz, thanks to Proposition 2.10 we can find a Lipschitz continuous version of $\partial_\mu h(x, \mu)$ as a function $x' \in \mathbb{R}^d \mapsto \partial_\mu h(x, \mu)(x')$.

2.3. **Non-local continuity and diffusion equations.**

We now provide a summary of the theory for the equations (1.2) and (1.3), based on [38, 39, 43]. We start by considering the Cauchy problem for the non-local continuity equation:

$$\begin{cases} \partial_t \mu_t + \operatorname{div}[b(t, x, \mu_t)\, \mu_t] = 0, \\ \mu|_{t=0} = \mu_0, \end{cases} \tag{2.7}$$

where the data of the problem are a fixed time horizon $T > 0$, a vector field $b : (0, T) \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and the initial probability measure $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$. The above equation has to be understood in the sense of distributions, yielding to the following definition.

Definition 2.12. Let $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$. A weak solution of (2.7) is a probability measure $\mu \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ such that

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t\varphi(t,x) + b(t,x,\mu_t)\cdot\nabla\varphi(t,x)\right)\mu_t(\,\mathrm{d}x)\,\mathrm{d}t = \int_{\mathbb{R}^d}\varphi(0,x)\mu_0(\,\mathrm{d}x),$$

for all test functions $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$.

We remark that $\mu \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ means that the map $t \in [0,T] \mapsto \mu_t \in \mathscr{P}_2(\mathbb{R}^d)$ is continuous with respect to the weak convergence of measures, i.e. the map

$$t \in [0,T] \to \int_{\mathbb{R}^d} \varphi(x)\mu_t(\,\mathrm{d}x),$$

is continuous for every $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$. Moreover, Definition 2.12 makes sense if

$$b(t,x,\mu_t) \in L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\,\mathrm{d}\mu_t)).$$

We will always work with vector fields satisfying the following assumptions.

Assumptions (B)

(B1) The non-local velocity field $(t,x,\mu)\mapsto b(t,x,\mu)$ is measurable with respect to $t\in[0,T]$ and it is continuous in the $|\cdot|\times W_2$-topology with respect to $(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$.

(B2) There exists $M>0$ such that

$$|b(t,x,\mu)| \le M\left(1+|x|+\int_{\mathbb{R}^d}|y|\mu(\,\mathrm{d}y)\right), \tag{2.8}$$

for all times $t\in[0,T]$ and any $(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)$.

(B3) There exists a constant $L>0$ such that

$$|b(t,x,\mu)-b(t,y,\nu)| \le L\left(|x-y|+W_2(\mu,\nu)\right), \tag{2.9}$$

for all times $t\in[0,T]$ and for any $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_2(\mathbb{R}^d)$.

(B4) The vector field b is $C^{1,1}$ regular, where the Lipschitz continuity of the L-derivative is:

$$\mathbb{E}\left[|\partial_\mu b(t,x',\mu')(X')-\partial_\mu b(t,x,\mu)(X)|^2\right] \le L^2\left(|x-x'|^2+\mathbb{E}\left[|X-X'|^2\right]\right)$$

for all $t\in[0,T]$, $x,x'\in\mathbb{R}^d$ and $X,X'\in L^2(\Omega;\mathbb{R}^d)$ with law, respectively, μ,μ'.

Remark 2.13. *Both in* **Assumptions (B)**, *and in the following* **Assumptions (J)** *below, we denote by M a constant related to boundedness, and by L a constant related to Lipschitz continuity. In particular, the Lipschitz constant L plays a crucial role in Theorem 1.1, since we require $\lambda > \Lambda(T,L)$.*

Note that from (B3) we also have that

$$|\nabla b(t,x,\mu)| + |\partial_\mu b(t,x,\mu)(x')| \le L.$$

By assuming the above hypotheses, the continuity equation (1.2) admits a unique solution. We resume this well-posedness result in the following theorem, see [43].

Theorem 2.14. *Let $b:(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ be a vector field satisfying* (B1), (B2), (B3). *Then, for each $\mu_0\in\mathscr{P}_2(\mathbb{R}^d)$ there exists a unique solution $\mu\in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ of* (2.7). *Moreover, if $\operatorname{supp}\mu_0$ is compact, there exists a constant $r>0$ such that*

$$\operatorname{supp}\mu_t \subset B_r, \quad \textit{for all } t,s\in[0,T].$$

We now study the diffusion equation.

$$\begin{cases}\partial_t\mu_t^\varepsilon + \operatorname{div}\left[b(t,x,\mu_t^\varepsilon)\mu_t^\varepsilon\right] = \varepsilon\Delta\mu_t^\varepsilon,\\ \mu^\varepsilon|_{t=0} = \mu_0.\end{cases} \tag{2.10}$$

First of all, a solution is a family of measures which satisfies the following.

Definition 2.15. Let $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$. A weak solution of (2.10) is a probability measure $\mu^\varepsilon \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ such that

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t\varphi(t,x) + b(t,x,\mu_t^\varepsilon)\cdot\nabla\varphi(t,x) + \varepsilon\Delta\varphi(t,x)\right)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t = \int_{\mathbb{R}^d}\varphi(0,x)\mu_0(\,\mathrm{d}x),$$

for all test functions $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$.

By assuming the same regularity on the vector field, we have the following existence and uniqueness theorem for (2.10), see [38, 39].

Theorem 2.16. *Let $b:(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ be a vector field which satisfies* (B1), (B2), (B3). *Then, for each $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a unique solution $\mu \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ of* (1.3).

We conclude this section with two technical lemmas.

Lemma 2.17. *Let b be a vector field satisfying* **Assumptions (B)**. *Then,*

$$\begin{aligned}|b(t,x',\mu') - b(t,x,\mu) - \nabla b(t,x,\mu)\cdot(x'-x) - \mathbb{E}\left[\partial_\mu b(t,x,\mu)(X)\cdot(X'-X)\right]|\\ \le \tfrac{L}{2}|x'-x|^2 + \tfrac{L}{2}\mathbb{E}[|X'-X|^2],\end{aligned} \tag{2.11}$$

for a.e. $t\in[0,T]$, for any $x,x'\in\mathbb{R}^d$ and $X,X'\in L^2(\Omega;\mathbb{R}^d)$ with law, respectively, μ,μ'. In particular, we have that

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^T \Big|b(t,X_t',\mathcal{L}(X_t')) - b(t,X_t,\mathcal{L}(X_t)) - \nabla b(t,X_t,\mathcal{L}(X_t))\cdot(X_t'-X_t)\\ -\tilde{\mathbb{E}}\left[\partial_\mu b(t,X_t,\mathcal{L}(X_t))(\tilde X_t)\cdot(\tilde X_t'-\tilde X_t)\right]\Big|\,\mathrm{d}t\Bigg] \le L\mathbb{E}\left[\int_0^T |X_t'-X_t|^2\,\mathrm{d}t\right],\end{aligned} \tag{2.12}$$

for any square integrable process $X_t, X_t' \in L^2((0,T);L^2(\Omega;\mathbb{R}^d))$

Proof. We add and subtract the quantity $b(t,x',\mu)$ in the absolute value on the left hand side. From the identity

$$b(t,x',\mu) - b(t,x,\mu) = \int_0^1 \nabla b(t,sx'+(1-s)x,\mu)\cdot(x'-x)\,\mathrm{d}s,$$

it easily follows that

$$\begin{aligned}&|b(t,x',\mu) - b(t,x,\mu) - \nabla b(t,x,\mu)\cdot(x'-x)|\\ &\qquad= \left|\int_0^1 \nabla b(t,sx'+(1-s)x,\mu)\cdot(x'-x)\,\mathrm{d}s - \nabla b(t,x,\mu)\cdot(x'-x)\right|\\ &\qquad\le \int_0^1 |\nabla b(t,sx'+(1-s)x,\mu) - \nabla b(t,x,\mu)||x'-x|\,\mathrm{d}s\\ &\qquad\le L|x'-x|^2\int_0^1 s\,\mathrm{d}s \le \frac{L}{2}|x'-x|^2.\end{aligned}$$

On the other hand, from the identity

$$b(t,x,\mu') - b(t,x,\mu) = \mathbb{E}\left[\int_0^1 \partial_\mu b(t,x,\mathcal{L}(sX'+(1-s)X)(sX'+(1-s)X)\cdot(X'-X)\,\mathrm{d}s\right],$$

it follows that

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 \partial_\mu b(t,x,\mathcal{L}(sX'+(1-s)X)(sX'+(1-s)X)\cdot(X'-X)\,\mathrm{d}s - \partial_\mu b(t,x,\mathcal{L}(X))(X)\cdot(X'-X)\right]\\ &\quad\le \mathbb{E}\left[\int_0^1 \left|\partial_\mu b(t,x,\mathcal{L}(sX'+(1-s)X)(sX'+(1-s)X) - \partial_\mu b(t,x,\mathcal{L}(X))(X)\right||X'-X|\,\mathrm{d}s\right]\\ &\quad\le \frac{L}{2}\mathbb{E}[|X'-X|^2].\end{aligned}$$

Then, the conclusion follows from the triangle inequality and the two estimates above. Finally, (2.12) follows from (2.11). □

We now recall the Osgood's lemma, see [25, 26].

Lemma 2.18. *Let ρ be a positive Borel function, γ a locally integrable positive function, ψ a continuous increasing strictly positive function, and $\eta > 0$. Assume that the function ρ satisfies one between*

$$\rho(t) \leq \eta + \int_{t_0}^{t} \gamma(s)\psi(\rho(s))\,\mathrm{d}s, \qquad \text{or} \qquad \rho(t) \leq \eta + \int_{t}^{t_0} \gamma(s)\psi(\rho(s))\,\mathrm{d}s. \tag{2.13}$$

Then it holds that

$$-\mathfrak{M}(\rho(t)) + \mathfrak{M}(\eta) \leq \int_{t}^{t_0} \gamma(s)\,\mathrm{d}s, \qquad \text{with } \mathfrak{M}(x) = \int_{x}^{1} \frac{1}{\psi(s)}\,\mathrm{d}s.$$

Proof. A proof of the lemma when ρ satisfies the first inequality in (2.13) can be found in [25]. We prove the lemma in the case ρ satisfies the second inequality in (2.13), which is stated but not proved in [26]. Define the function $R_\eta(t) := \eta + \int_{t}^{t_0} \gamma(s)\psi(\rho(s))\,\mathrm{d}s$. That implies $R_\eta(t) \geq \rho(t)$ by assumption. Since R is absolutely continuous, then it holds

$$\dot{R}_\eta(t) = -\gamma(t)\psi(\rho(t)) \geq -\gamma(t)\psi(R_\eta(t)) \quad \text{for a.e. } t,$$

and integrating the above expression in time

$$\int_{t}^{t_0} \frac{\dot{R}_\eta(s)}{\psi(R_\eta(s))}\,\mathrm{d}s \geq -\int_{t}^{t_0} \gamma(s)\,\mathrm{d}s.$$

Then, by the change of variables $s \to R_\eta(s)$ in the left hand side, we get that

$$\int_{R_\eta(t)}^{\eta} \frac{\mathrm{d}s}{\psi(s)} \geq -\int_{t}^{t_0} \gamma(s)\,\mathrm{d}s.$$

By the definition of $\mathfrak{M}$, the left hand side coincides with $\mathfrak{M}(\eta) - \mathfrak{M}(R_\eta(t))$. Then, by using that $\mathfrak{M}$ is decreasing we obtain that

$$-\mathfrak{M}(\rho(t)) + \mathfrak{M}(\eta) \leq -\mathfrak{M}(R_\eta(t)) + \mathfrak{M}(\eta) \leq \int_{t}^{t_0} \gamma(s)\,\mathrm{d}s$$

and this concludes the proof. □

3. Stochastic mean-field optimal control

In this section we provide a short overview of stochastic control theory. After introducing the notations, we will give an appropriate version of the Pontryagin Maximum Principle based on the tools introduced in Section 2. Most of the results are taken from [20, 21] and slightly modified to fit our context.

3.1. **The stochastic set-up.** Given a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, equipped with an adapted Brownian motion W_t, we denote by X_t^ε a stochastic process solving the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = (b(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon)) + \alpha_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t, \\ X_0^\varepsilon = \xi, \end{cases} \tag{3.1}$$

where $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ is a given random variable with $\mathcal{L}(\xi) = \mu_0$, and the admissible control α_t satisfies:

Assumption (A)

(A1) α_t is a measurable process with values in U. Define $R := \max(1, \max_{x\in U} |x|)$.

In particular, since U is compact, the following bound is trivial

$$\mathbb{E}\int_0^T |\alpha_t|^2\,\mathrm{d}t < \infty. \tag{3.2}$$

We now state a classical well-posedness result for (3.1), see [21].

Theorem 3.1. *If b satisfies* **Assumptions (B)** *and α_t satisfies* (3.2)*, then there exists a unique solution X_t^ε of* (3.1) *such that*

$$\mathbb{E}\left[\sup_{t\in(0,T)}|X_t^\varepsilon|^2\right] < \infty.$$

Moreover, if $X_t^{\varepsilon,'}$ is a solution of (3.1) *with control α_t' and initial condition ξ', then the following stability estimate holds*

$$\mathbb{E}\left[\sup_{t\in(0,T)}|X_t^{\varepsilon,'}-X_t^\varepsilon|^2\right] \le C_2(T,L)\left(\mathbb{E}\left[|\xi'-\xi|^2\right]+\mathbb{E}\left[\int_0^T|\alpha_t'-\alpha_t|^2\,\mathrm{d}t\right]\right), \tag{3.3}$$

where L is the Lipschitz constant of b in (B3)*.*

It is worth to remark that in Theorem 3.1 the existence is understood in the *strong* sense, i.e. one can find a solution to (3.1) on any given filtered probability space equipped with any given adapted Brownian motion. Moreover, *pathwise uniqueness* holds: it means that, on any given filtered probability space equipped with any given Brownian motion, any two solutions to (3.1) with the same initial condition ξ coincide. Since it will be crucial for the following, we compute the constants appearing in Theorem 3.1 under our specific assumptions. We recall that $M_2(\mu_0)$ is the second moment defined in (2.2).

Lemma 3.2. *Under the hypothesis of Theorem 3.1, for $\varepsilon\le 1$, we have that*

$$\mathbb{E}\left[\sup_{t\in(0,T)}|X_t^\varepsilon|^2\right] \le C_1(T,\mu_0,M,R),$$

with

$$C_1(T,\mu_0,M,R) := \left(\frac{M+R}{M+1}+\sqrt{M_2(\mu_0)+T}\right)^2 e^{(M+1)T}, \tag{3.4}$$

where M is the constant in (B2) *and R as in* (A1)*. Moreover, the constant C_2 in* (3.3) *is*

$$C_2(T,L) := \exp\{(4L+1)T\}. \tag{3.5}$$

Proof. We divide the proof in two steps.

Step 1 **Uniform bound on the forward component.** Let X_t^ε be the solution of (3.1) with control α_t and initial datum ξ_0. By an easy application of Ito's Lemma, we get that

$$\mathbb{E}[|X_t^\varepsilon|^2] = \mathbb{E}[|\xi|^2] + \mathbb{E}\left[\int_0^T (b(t,X_t^\varepsilon,\mathcal{L}(X_t^\varepsilon))+\alpha_t)\cdot X_t^\varepsilon\,\mathrm{d}t\right] + \varepsilon t.$$

Notice that $|\alpha_t|\le R$ for all $t\in[0,T]$, since it takes values in the compact set U. Then, by using the growth assumptions (B2) on b, L^∞ bound on α_t, Lemma 2.18, and Doob's inequality [37] we get

$$\mathbb{E}\left[\sup_{t\in(0,T)}|X_t^\varepsilon|^2\right] \le \left(\frac{M+R}{M+1}+\sqrt{M_2(\mu_0)+T}\right)^2 e^{(M+1)T}. \tag{3.6}$$

Step 2 **Stability estimate for the forward component.** Let $X_t^\varepsilon, X_t^{\varepsilon,'}$ be, respectively, the solutions of (3.1) with control α_t,α_t' and initial condition ξ,ξ'. Then, the difference $X_t^{\varepsilon,'}-X_t^\varepsilon$ satisfies the ordinary differential equation

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}(X_t^{\varepsilon,'}-X_t^\varepsilon) = b(t,X_t^{\varepsilon,'},\mathcal{L}(X_t^{\varepsilon,'}))+\alpha_t'-b(t,X_t^\varepsilon,\mathcal{L}(X_t^\varepsilon))-\alpha_t,\\ X_0^{\varepsilon,'}-X_0^\varepsilon = \xi'-\xi,\end{cases} \tag{3.7}$$

and by using the Lipschitz property (B3) and standard estimates we get that

$$\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right] \le \exp\{(4L+1)T\}\left(\mathbb{E}\left[|\xi' - \xi|^2 + \int_0^T |\alpha_t' - \alpha_t|^2 \,\mathrm{d}t\right]\right). \tag{3.8}$$

□

We now proceed by defining the following stochastic optimal control problem.

Stochastic optimal control problem (SOC)

Denote by $\boldsymbol{\alpha} := (\alpha_t)_{0\le t\le T}$ the control on the whole time interval, and consider the cost functional

$$J^S(\boldsymbol{\alpha}) = \mathbb{E}\left[g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon)) + \int_0^T (f(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon)) + \psi(\alpha_t))\,\mathrm{d}t\right]. \tag{3.9}$$

Find

$$\min_{\boldsymbol{\alpha}} J^S(\boldsymbol{\alpha}),$$

such that X_t^ε is a solution of (3.1) and the control satisfies **Assumption (A)**.

From now on, we assume the following hypothesis on the cost J.

Assumptions (J)

(J1) The control cost $\alpha \in U \to \psi(\alpha) \in \mathbb{R}$ is C^1.

(J2) The functions f and g are C^1: in particular, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathscr{P}_2(\mathbb{R}^d)$ it holds

$$|f(t,x,\mu) - f(t,x',\mu')| \le M\left[1 + |x| + |x'| + M_2(\mu) + M_2(\mu')\right]\left(|x - x'| + W_2(\mu,\mu')\right),$$
$$|g(x,\mu) - g(x',\mu')| \le M\left[1 + |x| + |x'| + M_2(\mu) + M_2(\mu')\right]\left(|x - x'| + W_2(\mu,\mu')\right).$$

(J3) The derivatives of f and g with respect to x are L-Lipschitz continuous with respect to (x,μ), i.e. for every $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, and $\mu, \mu' \in \mathscr{P}_2(\mathbb{R}^d)$ it holds

$$|\nabla_x f(t,x,\mu) - \nabla_x f(t,x',\mu')| \le L\left(|x - x'| + W_2(\mu,\mu')\right),$$
$$|\nabla_x g(x,\mu) - \nabla_x g(x',\mu')| \le L\left(|x - x'| + W_2(\mu,\mu')\right),$$

and for all $X, X' \in L^2(\Omega;\mathbb{R}^d)$ with law, respectively, μ, μ' it holds

$$\mathbb{E}\left[|\partial_\mu f(t,x',\mu')(X') - \partial_\mu f(t,x,\mu)(X)|^2\right] \le L^2\left(|x - x'|^2 + \mathbb{E}\left[|X - X'|^2\right]\right),$$

$$\mathbb{E}\left[|\partial_\mu g(x',\mu')(X') - \partial_\mu g(x,\mu)(X)|^2\right] \le L^2\left(|x - x'|^2 + \mathbb{E}\left[|X - X'|^2\right]\right).$$

Note that from (J3) we also have that the derivatives of f and g are bounded, i.e for every $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, and $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ it holds

$$|\nabla_x f(t,x,\mu)| + |\nabla_x g(x,\mu)| \le L, \qquad |\partial_\mu f(t,x,\mu)(x')| + |\partial_\mu g(x,\mu)(x')| \le L. \tag{3.10}$$

We associate to (SOC) the Hamiltonian:

$$H(t,x,\mu,y,\alpha) = (b(t,x,\mu) + \alpha)\cdot y + f(t,x,\mu) + \psi(\alpha), \tag{3.11}$$

for $(t,x,\mu,y,\alpha) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times U$. It is worth to note that under **Assumptions (A), (B), (J)**, the Hamiltonian H is $C^{1,1}_{\mathrm{loc}}$-regular.

In full analogy with the Pontryagin Maximum Principle for finite-dimensional control problems, we introduce an adjoint process as the solution of a backward equation involving partial derivatives of the Hamiltonian with respect to the measure argument. Then, for a given admissible control α_t and the corresponding controlled state X_t^ε, we give the following definition.

Definition 3.3. We call *adjoint processes* of X_t^ε any couple $(Y_t^\varepsilon, Z_t^\varepsilon)$ satisfying the backward stochastic equation

$$\begin{cases} \mathrm{d}Y_t^\varepsilon = -\left[\nabla_x H(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon), Y_t^\varepsilon, \alpha_t) + \tilde{\mathbb{E}}[\partial_\mu H(t, \tilde{X}_t^\varepsilon, \mathcal{L}(\tilde{X}_t^\varepsilon), \tilde{Y}_t^\varepsilon, \tilde{\alpha}_t)(X_t^\varepsilon)]\right] \mathrm{d}t + Z_t^\varepsilon \, \mathrm{d}W_t, \\ Y_T^\varepsilon = \nabla g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon)) + \tilde{\mathbb{E}}\left[\partial_\mu g(\tilde{X}_T^\varepsilon, \mathcal{L}(\tilde{X}_T^\varepsilon))(X_T^\varepsilon)\right], \end{cases} \tag{3.12}$$

where $(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon, \tilde{\alpha}_t)$ is an independent copy of $(X_t^\varepsilon, Y_t^\varepsilon, \alpha_t)$ defined on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$. In particular, it holds $\mathcal{L}(\tilde{X}_t^\varepsilon) = \mathcal{L}(X_t^\varepsilon)$.

A solution of (3.12) is a couple $(Y_t^\varepsilon, Z_t^\varepsilon)$, where the introduction of the process Z_t^ε is necessary to ensure that the process Y_t^ε is adapted with respect to the *forward filtration* $\mathcal{F}_t$.

It is worth to note that the functions $\nabla_x H$ and $\partial_\mu H$ do not depend on α, due to the particular form of the Hamiltonian (3.11). As a consequence, the system can be rewritten as

$$\begin{cases} \mathrm{d}Y_t^\varepsilon & = -\left[\nabla_x b(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon))Y_t^\varepsilon + \nabla_x f(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon))\right] \mathrm{d}t \\ & \quad - \left[\tilde{\mathbb{E}}[\partial_\mu b(t, \tilde{X}_t^\varepsilon, \mathcal{L}(\tilde{X}_t^\varepsilon))(X_t^\varepsilon)\tilde{Y}_t^\varepsilon + \partial_\mu f(t, \tilde{X}_t^\varepsilon, \mathcal{L}(\tilde{X}_t^\varepsilon)(X_t^\varepsilon)]\right] \mathrm{d}t + Z_t^\varepsilon \, \mathrm{d}W_t, \\ Y_T^\varepsilon & = \nabla g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon)) + \tilde{\mathbb{E}}\left[\partial_\mu g(\tilde{X}_T^\varepsilon, \mathcal{L}(\tilde{X}_T^\varepsilon))(X_T^\varepsilon)\right], \end{cases} \tag{3.13}$$

The equation (3.12) is a *backward stochastic differential equation* (BSDE) of mean-field type, since the law of Y_t^ε appears in the term which involves the L-derivative of H. This kind of BSDE admits a unique solution, if we assume enough regularity on the coefficients and we consider $X_t^\varepsilon, \alpha_t$ as given data of the problem. In particular, the following theorem holds, see [21].

Theorem 3.4. *Let α_t be an admissible control and X_t^ε the corresponding trajectory. Under* **Assumptions (A), (B), (J)***, there exists a unique solution $(Y_t^\varepsilon, Z_t^\varepsilon)$ such that*

$$\mathbb{E}\left[\sup_{t\in(0,T)} |Y_t^\varepsilon|^2 + \int_0^T |Z_t^\varepsilon|^2 \, \mathrm{d}t\right] < \infty. \tag{3.14}$$

Moreover, if $(Y_t^{\varepsilon,'}, Z_t^{\varepsilon,'})$ is a solution corresponding to a control α_t' and a stochastic process $X_t^{\varepsilon,'}$, it holds that

$$\mathbb{E}\left[\sup_{t\in(0,T)} |Y_t^{\varepsilon,'} - Y_t^\varepsilon|^2 + \int_0^T |Z_t^{\varepsilon,'} - Z_t^\varepsilon|^2 \, \mathrm{d}t\right] \leq C(T,L)\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right], \tag{3.15}$$

where L is the Lipschitz constant appearing in **Assumptions (B), (J)**.

Similarly to what we have done for the forward component, we explicitly compute the constants ensuring boundedness of Y_t^ε and well-posedness of solutions of (3.13).

Lemma 3.5. *Let α_t be an admissible control and X_t^ε the corresponding trajectory. Under the assumptions of Theorem 3.4, for every $p \geq 1$ it holds*

$$\mathbb{E}\left[\sup_{t\in(0,T)} |Y_t^\varepsilon|^p\right] \leq C_3(T,L)^p, \tag{3.16}$$

where

$$C_3(T,L) := (1+2L)e^{2LT}. \tag{3.17}$$

Finally, given another trajectory $X_t^{\varepsilon,'}$, we have that

$$\mathbb{E}\left[\sup_{t\in(0,T)} |Y_t^{\varepsilon,'} - Y_t^\varepsilon|^2\right] \leq C_4(T,L)\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right] \tag{3.18}$$

and the constant $C_4(T,L)$ is given by

$$C_4(T,L) := 4L^2(2 + T + TC_3(T,L)^2)e^{(6+2L^2)T}. \tag{3.19}$$

Proof. We divide the proof in two steps.

Step 1 **Uniform bounds on Y_t^ε.** Let $p \geq 2$, an application of Ito's lemma gives that

$$\begin{aligned}\mathbb{E}\left[|Y_t^\varepsilon|^p\right] &+ p(p-1)\mathbb{E}\left[\int_t^T |Z_s^\varepsilon|^2 |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] = \mathbb{E}\left[|Y_T^\varepsilon|^p\right] \\ &+ p\mathbb{E}\left[\int_t^T (\nabla_x b(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))Y_s^\varepsilon + \nabla_x f(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))) \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] \\ &+ p\mathbb{E}\left[\int_t^T \left(\tilde{\mathbb{E}}[\partial_\mu b(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon))(X_s^\varepsilon)\tilde{Y}_s^\varepsilon + \partial_\mu f(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon)(X_s^\varepsilon)]\right) \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right].\end{aligned}$$

We estimate the terms on the right hand side separately. First, by (3.10) we have that

$$\mathbb{E}\left[|Y_T^\varepsilon|^p\right] \leq (2L)^p. \tag{3.20}$$

We now consider the term involving $\nabla_x b$: by using assumption (B3) we easily obtain that

$$\mathbb{E}\left[\int_t^T \nabla_x b(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))Y_s^\varepsilon \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] \leq L\int_t^T \mathbb{E}\left[|Y_s^\varepsilon|^p\,\mathrm{d}s\right]. \tag{3.21}$$

On the other hand, for the part involving $\partial_\mu b$ we use assumption (B4) and Holder's inequality, obtaining

$$\begin{aligned}\mathbb{E}\left[\int_t^T \tilde{\mathbb{E}}\big[\partial_\mu b(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon))(X_s^\varepsilon)\tilde{Y}_s^\varepsilon\big] \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] &\leq L\int_t^T \tilde{\mathbb{E}}[|Y_s^\varepsilon|]\mathbb{E}[|Y_s^\varepsilon|^{p-1}]\,\mathrm{d}s \\ &\leq L\int_t^T \mathbb{E}[|Y_s^\varepsilon|^p]^{\frac{1}{p}}\mathbb{E}[|Y_s^\varepsilon|^p]^{\frac{p-1}{p}}\,\mathrm{d}s = L\int_t^T \mathbb{E}[|Y_s^\varepsilon|^p]\,\mathrm{d}s.\end{aligned} \tag{3.22}$$

Finally, for the terms involving f we use (3.10) to get

$$\mathbb{E}\left[\int_t^T \nabla_x f(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon)) \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] \leq L\int_t^T \mathbb{E}\left[|Y_s^\varepsilon|^p\right]^{\frac{p-1}{p}}\,\mathrm{d}s, \tag{3.23}$$

$$\mathbb{E}\left[\int_t^T \tilde{\mathbb{E}}\left[\partial_\mu f(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon)(X_s^\varepsilon)\right] \cdot Y_s^\varepsilon |Y_s^\varepsilon|^{p-2}\,\mathrm{d}s\right] \leq L\int_t^T \mathbb{E}\left[|Y_s^\varepsilon|^p\right]^{\frac{p-1}{p}}\,\mathrm{d}s. \tag{3.24}$$

Putting together the previous estimates, we obtain

$$\mathbb{E}\left[|Y_t^\varepsilon|^p\right] \leq (2L)^p + 2Lp\int_t^T \mathbb{E}[|Y_s^\varepsilon|^p] + \mathbb{E}\left[|Y_s^\varepsilon|^p\right]^{\frac{p-1}{p}}\,\mathrm{d}s. \tag{3.25}$$

By defining $y(t) := \mathbb{E}\left[|Y_t^\varepsilon|^p\right]$, the above inequality can be rewritten as

$$y(t) \leq (2L)^p + 2Lp\int_t^T y(s) + y(s)^{\frac{p-1}{p}}\,\mathrm{d}s, \tag{3.26}$$

and an application of Lemma 2.18 provides the following estimate

$$y(t)^{1/p} \leq (1 + 2L)e^{2LT}. \tag{3.27}$$

In other words, we get that

$$\|Y_t^\varepsilon\|_{L^p(\Omega)} \leq (1 + 2L)e^{2LT}. \tag{3.28}$$

The same bound trivially holds for $1 \leq p < 2$ since we are working on a probability space. Moreover, it is a classical fact that, since the right hand side of (3.28) is uniformly bounded in p, then $Y_t^\varepsilon \in L^\infty(\Omega)$ (see [48, Exercise 1.3.5]) and moreover

$$\sup_{t\in(0,T)}\sup_{\omega\in\Omega} |Y_t^\varepsilon| \leq (1 + 2L)e^{2LT}. \tag{3.29}$$

<u>*Step 2*</u> **Stability estimate on the backward component Y_t^ε.** Let $X_t^\varepsilon, X_t^{\varepsilon,'}$ be, respectively, the solutions of (3.1) with control α_t, α_t' and initial condition ξ, ξ'. We denote with $Y_t^\varepsilon, Y_t^{\varepsilon,'}$ the associate adjoint processes. Applying Ito's lemma to $|Y_t^{\varepsilon,'} - Y_t^\varepsilon|^2$ we get that

$$
\begin{aligned}
\mathbb{E}\left[|Y_t^{\varepsilon,'} - Y_t^\varepsilon|^2\right] &+ \mathbb{E}\left[\int_t^T |Z_t^{\varepsilon,'} - Z_t^\varepsilon|^2\,\mathrm{d}t\right] = \mathbb{E}\left[|Y_T^{\varepsilon,'} - Y_T^\varepsilon|^2\right] \\
&+2\mathbb{E}\left[\int_t^T \Big(\nabla_x b(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'}))Y_s^{\varepsilon,'} - \nabla_x b(t, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))Y_s^\varepsilon\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&+2\mathbb{E}\left[\int_t^T \Big(\nabla_x f(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'})) - \nabla_x f(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&+2\mathbb{E}\left[\int_t^T \Big(\tilde{\mathbb{E}}\left[\partial_\mu b(s, \tilde{X}_s^{\varepsilon,'}, \mathcal{L}(\tilde{X}_s^{\varepsilon,'}))(X_s^{\varepsilon,'})\tilde{Y}_s^{\varepsilon,'} - \partial_\mu b(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon))(X_s^\varepsilon)\tilde{Y}_s^\varepsilon\right]\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&+2\mathbb{E}\left[\int_t^T \Big(\tilde{\mathbb{E}}\left[\partial_\mu f(s, \tilde{X}_s^{\varepsilon,'}, \mathcal{L}(\tilde{X}_s^{\varepsilon,'}))(X_s^{\varepsilon,'}) - \partial_\mu f(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon))(X_s^\varepsilon)\right]\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right].
\end{aligned}
$$

We estimate the terms in the above inequality separately. First, by using (J3), for the part involving the final datum we have that

$$
\begin{aligned}
\mathbb{E}\left[|Y_T^{\varepsilon,'} - Y_T^\varepsilon|^2\right] \le& 2\mathbb{E}\left[|\nabla g(X_T^{\varepsilon,'}, \mathcal{L}(X_T^{\varepsilon,'})) - \nabla g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon))|^2\right] \\
&+ 2\mathbb{E}\left[\left|\tilde{\mathbb{E}}\left[\partial_\mu g(\tilde{X}_T^{\varepsilon,'}, \mathcal{L}(\tilde{X}_T^{\varepsilon,'}))(X_T^{\varepsilon,'}) - \partial_\mu g(\tilde{X}_T^\varepsilon, \mathcal{L}(\tilde{X}_T^\varepsilon))(X_T^\varepsilon)\right]\right|^2\right] \\
\le& 8L^2\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right].
\end{aligned}
$$

Second, for the part involving the running cost f, by using Young's inequality and (J3) we obtain

$$
\begin{aligned}
2\mathbb{E}&\left[\int_t^T \Big(\nabla_x f(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'})) - \nabla_x f(s, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&\le \mathbb{E}\left[\int_t^T |Y_s^{\varepsilon,'} - Y_s^\varepsilon|^2\,\mathrm{d}t\right] + 2L^2T\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right],
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
2\mathbb{E}&\left[\int_t^T \Big(\tilde{\mathbb{E}}\left[\partial_\mu f(s, \tilde{X}_s^{\varepsilon,'}, \mathcal{L}(\tilde{X}_s^{\varepsilon,'}))(X_s^{\varepsilon,'}) - \partial_\mu f(s, \tilde{X}_s^\varepsilon, \mathcal{L}(\tilde{X}_s^\varepsilon))(X_s^\varepsilon)\right]\Big) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&\le \mathbb{E}\left[\int_t^T |Y_s^{\varepsilon,'} - Y_s^\varepsilon|^2\,\mathrm{d}t\right] + 2L^2T\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,'} - X_t^\varepsilon|^2\right].
\end{aligned}
\tag{3.31}
$$

Last, we consider the part involving $\nabla_x b$ (the one which involves $\partial_\mu b$ works similar). We add and subtract the quantity $\nabla_x b(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'}))Y_s^\varepsilon$, then write

$$
\begin{aligned}
2\mathbb{E}&\left[\int_t^T (\nabla_x b(s, X_s^{\varepsilon,'},\ \mathcal{L}(X_s^{\varepsilon,'}))Y_s^{\varepsilon,'} - \nabla_x b(t, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))Y_s^\varepsilon) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&= 2\mathbb{E}\left[\int_t^T (\nabla_x b(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'})) - \nabla_x b(t, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon)))Y_s^\varepsilon \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&+2\mathbb{E}\left[\int_t^T \nabla_x b(s, X_s^{\varepsilon,'}, \mathcal{L}(X_s^{\varepsilon,'}))(Y_s^{\varepsilon,'} - Y_s^\varepsilon) \cdot (Y_s^{\varepsilon,'} - Y_s^\varepsilon)\,\mathrm{d}s\right]
\end{aligned}
$$

and by using Young's inequality we simply estimate as follows

$$
\begin{aligned}
2\mathbb{E}&\left[\int_t^T (\nabla_x b(s, X_s^{\varepsilon,\prime},\ \mathcal{L}(X_s^{\varepsilon,\prime}))Y_s^{\varepsilon,\prime} - \nabla_x b(t, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))Y_s^\varepsilon) \cdot (Y_s^{\varepsilon,\prime} - Y_s^\varepsilon)\,\mathrm{d}s\right] \\
&\le 2\mathbb{E}\left[\int_t^T |Y_s^{\varepsilon,\prime} - Y_s^\varepsilon|^2\,\mathrm{d}t\right] + \mathbb{E}\left[\int_t^T |\nabla_x b(s, X_s^{\varepsilon,\prime}, \mathcal{L}(X_s^{\varepsilon,\prime}))|^2 |Y_s^{\varepsilon,\prime} - Y_s^\varepsilon|^2\,\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_t^T |\nabla_x b(s, X_s^{\varepsilon,\prime}, \mathcal{L}(X_s^{\varepsilon,\prime})) - \nabla_x b(t, X_s^\varepsilon, \mathcal{L}(X_s^\varepsilon))|^2 |Y_s^\varepsilon|^2\,\mathrm{d}s\right] \\
&\le (2 + L^2)\mathbb{E}\left[\int_t^T |Y_s^{\varepsilon,\prime} - Y_s^\varepsilon|^2\,\mathrm{d}t\right] + 2TL^2 C_3(T,L)^2 \mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,\prime} - X_t^\varepsilon|^2\right],
\end{aligned}
$$

where we used (B4) and (3.29). By a Gronwall's type argument we get that

$$
\mathbb{E}\left[|Y_t^{\varepsilon,\prime} - Y_t^\varepsilon|^2\right] \le C_4(T,L)\mathbb{E}\left[\sup_{t\in(0,T)} |X_t^{\varepsilon,\prime} - X_t^\varepsilon|^2\right]
$$

for the constant $C_4(T,L)$ given in (3.19). □

We now provide the last set of assumptions, that are convexity hypotheses on the cost J.

Assumptions (C)

(C1) The control cost $\alpha \mapsto \psi(\alpha) \in \mathbb{R}$ is λ-convex over U, with convexity constant $\lambda > 0$, i.e. for all $\alpha, \alpha' \in U$ it holds

$$
\psi(\alpha') \ge \psi(\alpha) + \partial_\alpha \psi(\alpha) \cdot (\alpha' - \alpha) + \lambda|\alpha' - \alpha|^2.
$$

(C2) The running cost f is L-convex, i.e.

$$
f(t, x', \mu') \ge f(t, x, \mu) + \nabla_x f(t, x, \mu) \cdot (x' - x) + \mathbb{E}\left[\partial_\mu f(t, x, \mu)(X) \cdot (X' - X)\right],
$$

for a.e. $t \in [0,T]$, for every $x, x' \in \mathbb{R}^d$ and $X, X' \in L^2(\Omega; \mathbb{R}^d)$ with law, respectively, μ, μ'.

(C3) The final cost g is L-convex, i.e.

$$
g(x', \mu') \ge g(x, \mu) + \nabla_x g(x, \mu) \cdot (x' - x) + \mathbb{E}\left[\partial_\mu g(x, \mu)(X) \cdot (X' - X)\right],
$$

for every $x, x' \in \mathbb{R}^d$ and $X, X' \in L^2(\Omega; \mathbb{R}^d)$ with law, respectively, μ, μ'.

With the assumptions above, we can now prove the following sufficient condition on the control for optimality. The following theorem is a slight generalization of [20, Theorem 4.7].

Theorem 3.6. *Let $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, $\hat{\alpha}_t$ be an admissible control, X_t^ε the corresponding controlled state process, and $(Y_t^\varepsilon, Z_t^\varepsilon)$ the corresponding adjoint processes, and assume that* **Assumptions (A), (B), (C), (J)** *hold. Let $\Lambda(T,L) := TL(1+2L)e^{(6L+1)T}$. Then, if $\lambda > \Lambda(T,L)$ and moreover it holds $\mathcal{L}^1 \otimes \mathbb{P}$-a.e. that*

$$
H(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon), Y_t^\varepsilon, \hat{\alpha}_t) = \inf_{\alpha\in U} H(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon), Y_t^\varepsilon, \alpha),
$$

then $\hat{\alpha}_t$ is the unique optimal control, i.e. $J^S(\hat{\boldsymbol{\alpha}}) = \min_{\boldsymbol{\alpha}'} J^S(\boldsymbol{\alpha}')$ where the minimum is computed among the admissible controls.

Proof. We drop the ε superscript for simplicity of notations. Let α_t' be an admissible control and X' the associated controlled state. By computing the cost functional and using the definition of H

in (3.11), we have that

$$
\begin{aligned}
J^S(\hat{\boldsymbol{\alpha}}) - J^S(\boldsymbol{\alpha}') =& \mathbb{E}\left[g(X_T,\mathcal{L}(X_T)) - g(X'_T,\mathcal{L}(X'_T))\right] \\
&+ \mathbb{E}\left[\int_0^T (f(t,X_t,\mathcal{L}(X_t)) - f(t,X'_t,\mathcal{L}(X'_t)) + (\psi(\hat{\alpha}_t) - \psi(\alpha'_t))\,\mathrm{d}t\right] \\
=& \mathbb{E}\left[g(X_T,\mathcal{L}(X_T)) - g(X'_T,\mathcal{L}(X'_T))\right] \\
&+ \mathbb{E}\left[\int_0^T (H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t) - H(t,X'_t,\mathcal{L}(X'_t),Y_t,\alpha'_t))\,\mathrm{d}t\right] \\
&- \mathbb{E}\left[\int_0^T \big((b(t,X_t,\mathcal{L}(X_t)) - b(t,X'_t,\mathcal{L}(X'_t)) + \hat{\alpha}_t - \alpha'_t)\cdot Y_t\big)\,\mathrm{d}t\right].
\end{aligned}
\tag{3.32}
$$

We estimate the terms involving the final cost, by using convexity of g:

$$
\begin{aligned}
\mathbb{E}\left[g(X_T,\mathcal{L}(X_T)) - g(X'_T,\mathcal{L}(X'_T))\right] \le& \mathbb{E}\left[\nabla_x g(X_T,\mathcal{L}(X_T))\cdot(X_T - X'_T)\right] \\
&+ \mathbb{E}\tilde{\mathbb{E}}\left[\partial_\mu g(X_T,\mathcal{L}(X_T))(\tilde{X}_T)\cdot(\tilde{X}_T - \tilde{X}'_T)\right].
\end{aligned}
$$

By using Fubini's Theorem, the fact that the tilde random variables are independent copies of the non-tilde variables, and the definition of Y_T, we have

$$
\mathbb{E}\left[g(X_T,\mathcal{L}(X_T)) - g(X'_T,\mathcal{L}(X'_T))\right] \le \mathbb{E}\left[Y_T\cdot(X_T - X'_T)\right]. \tag{3.33}
$$

We now use the adjoint equation to compute

$$
\begin{aligned}
\mathbb{E}\left[Y_T\cdot(X_T - X'_T)\right] =& \mathbb{E}\left[\int_0^T (X_t - X'_t)\cdot\,\mathrm{d}Y_t + \int_0^T Y_t\cdot\,\mathrm{d}(X_t - X'_t)\right] \\
=& -\mathbb{E}\left[\int_0^T \nabla_x H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t)\cdot(X_t - X'_t)\,\mathrm{d}t\right] \\
&- \mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left[\partial_\mu H(t,\tilde{X}_t,\mathcal{L}(\tilde{X}_t),\tilde{Y}_t,\tilde{\hat{\alpha}}_t)(X_t)\right]\cdot(X_t - X'_t)\,\mathrm{d}t\right] \\
&+ \mathbb{E}\left[\int_0^T \big(b(t,X_t,\mathcal{L}(X_t)) - b(t,X'_t,\mathcal{L}(X'_t)) + \hat{\alpha}_t - \alpha'_t)\big)\cdot Y_t\,\mathrm{d}t\right].
\end{aligned}
\tag{3.34}
$$

Again by Fubini's theorem, it holds

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left[\partial_\mu H(t,\tilde{X}_t,\mathcal{L}(\tilde{X}_t),\tilde{Y}_t,\tilde{\hat{\alpha}}_t)(X_t)\right]\cdot(X_t - X'_t)\,\mathrm{d}t\right] \\
&= \mathbb{E}\tilde{\mathbb{E}}\left[\int_0^T \partial_\mu H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t)(\tilde{X}_t)\cdot(\tilde{X}_t - \tilde{X}'_t)\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left[\partial_\mu H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t)(\tilde{X}_t)\right]\cdot(\tilde{X}_t - \tilde{X}'_t)\,\mathrm{d}t\right].
\end{aligned}
\tag{3.35}
$$

Then, by using (3.32), (3.33), (3.34), and (3.35) we have

$$
\begin{aligned}
J^S(\hat{\boldsymbol{\alpha}}) - J^S(\boldsymbol{\alpha}') \le& \mathbb{E}\left[\int_0^T (H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t) - H(t,X'_t,\mathcal{L}(X'_t),Y_t,\alpha'_t))\,\mathrm{d}t\right] \\
&- \mathbb{E}\left[\int_0^T \nabla_x H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t)\cdot(X_t - X'_t)\,\mathrm{d}t\right] \\
&- \mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left[\partial_\mu H(t,X_t,\mathcal{L}(X_t),Y_t,\hat{\alpha}_t))(\tilde{X}_t)\cdot(\tilde{X}_t - \tilde{X}'_t)\right]\,\mathrm{d}t\right].
\end{aligned}
\tag{3.36}
$$

We estimate the right hand side of (3.36) as follows: by **Assumptions (C)**, for the part involving the running cost we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T f(t,X_t',\mathcal{L}(X_t'))\,\mathrm{d}t\right] \geq \mathbb{E}&\left[\int_0^T f(t,X_t,\mathcal{L}(X_t))\,\mathrm{d}t\right]\\ &+\mathbb{E}\left[\int_0^T \nabla f(t,X_t,\mathcal{L}(X_t))\cdot(X_t'-X_t)\,\mathrm{d}t\right]\\ &+\mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left[\partial_\mu f(t,X_t,\mathcal{L}(X_t))(\tilde{X}_t)\cdot(\tilde{X}_t'-\tilde{X}_t)\right]\,\mathrm{d}t\right].\end{aligned}$$

Then, we estimate the part involving the vector field b: we apply Lemma 2.17, Lemma 3.2 and Lemma 3.5 to obtain

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^T Y_t\cdot\Bigg(b(t,X_t,\mathcal{L}(X_t))-b(t,X_t',\mathcal{L}(X_t'))-\nabla b(t,X_t,\mathcal{L}(X_t))(X_t-X_t')\Bigg)\,\mathrm{d}t\Bigg]&\\ -\mathbb{E}\Bigg[\int_0^T Y_t\cdot\Bigg(\tilde{\mathbb{E}}\Bigg[\partial_\mu b(t,X_t,\mathcal{L}(X_t))(\tilde{X}_t)(\tilde{X}_t-\tilde{X}_t')\Bigg]\Bigg)\,\mathrm{d}t\Bigg]&\\ \leq \Lambda(T,L)\left(\mathbb{E}\left[\int_0^T|\hat{\alpha}_t-\alpha_t'|^2\,\mathrm{d}t\right]\right),&\end{aligned}$$

where

$$\Lambda(T,L) := C_2(T,L)C_3(T,L)LT. \tag{3.37}$$

Finally, for the part involving the control cost, thanks to minimality of $\hat{\alpha}_t$ and convexity of ψ, it holds

$$\mathbb{E}\left[\int_0^T \psi(\hat{\alpha}_t)-\psi(\alpha_t')+(\hat{\alpha}_t-\alpha_t')\cdot Y_t\,\mathrm{d}t\right]\leq -\lambda\mathbb{E}\int_0^T|\hat{\alpha}_t-\alpha_t'|^2\,\mathrm{d}t. \tag{3.38}$$

In conclusion, we have obtained that

$$J^S(\hat{\boldsymbol{\alpha}})+(\lambda-\Lambda(T,L))\mathbb{E}\left[\int_0^T|\hat{\alpha}_t-\alpha_t'|^2\,\mathrm{d}t\right]\leq J^S(\boldsymbol{\alpha}'), \tag{3.39}$$

which in turn gives that $\hat{\boldsymbol{\alpha}}$ is the unique optimal control if $\lambda > \Lambda(T,L)$. □

Remark 3.7. *Note that, if b is affine (eventually depending on the barycenter of μ too), i.e. of the form*

$$b(t,x,\mu)=b_0(t)+b_1(t)x+b_2(t)\int_{\mathbb{R}^d}x\,\mu(\,\mathrm{d}x),$$

then **Assumption (C)** *implies that the function $(x,\mu,\alpha)\mapsto H(t,x,\mu,y,\alpha)$ is convex. The proof of Theorem 3.6 follows then in a simpler way, without resorting to Lemma 2.17, see* [21].

Remark 3.8. *Note that the constant $\Lambda(T,L)$ can be made as small as desired for T small enough. This implies that, at least for small times, the condition $\lambda>\Lambda(T,L)$ is always satisfied. A similar condition also ensures Lipschitz regularity of mean-field optimal control (see* [15]*) and uniqueness of a minimizer in mean-field games (see* [5]*).*

Remark 3.9. *It is clear from the proof of Theorem 3.6 that the convexity hypothesis on f and g can be dropped at the cost of requiring a larger lower bound for the constant λ. This is indeed the case of* [15] *where the running and final cost are not required to be convex. On the other hand, strict convexity of f and g would allow to consider a larger class of control costs ψ.*

We now show that the optimal control is Lipschitz continuous when the Hamiltonian is strictly convex with respect to the control.

Lemma 3.10. *Under* **Assumptions (A), (B), (C), (J)***, there exists a unique minimizer $\hat{\alpha}$ of H. Moreover, the function $\hat{\alpha}: y\in\mathbb{R}^d\to\hat{\alpha}(y)\in U$ is measurable and Lipschitz continuous, with a Lipschitz constant depending on λ only.*

Proof. Observe that, for any (t, x, μ, y), the function $\alpha \mapsto H(t, x, \mu, y, \alpha)$ is continuously differentiable and strictly convex, so that $\hat{\alpha}(y)$ appears as the unique solution of the variational inequality:

$$\forall \beta \in U, \qquad \text{it holds} \qquad (\hat{\alpha}(y) - \beta) \cdot (y + \nabla\psi(\hat{\alpha})) \leq 0. \tag{3.40}$$

Moreover, by strict convexity, measurability of $\hat{\alpha}(y)$ is a consequence of the *gradient descent* algorithm with convex constraint, see [21]. We now prove Lipschitz continuity: for any $y, y' \in \mathbb{R}^d$ and $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, the criticality of $\hat{\alpha}$ provides the following inequalities

$$\big(\hat{\alpha}(y) - \hat{\alpha}(y')\big) \cdot \partial_\alpha H(t, x, \mu, y', \hat{\alpha}(y')) \geq 0, \tag{3.41}$$

$$\big(\hat{\alpha}(y') - \hat{\alpha}(y)\big) \cdot \partial_\alpha H(t, x, \mu, y, \hat{\alpha}(y)) \geq 0. \tag{3.42}$$

They in turn imply

$$\big(\hat{\alpha}(y') - \hat{\alpha}(y)\big) \cdot \big(\partial_\alpha H(t, x, \mu, y', \hat{\alpha}(y')) - \partial_\alpha H(t, x, \mu, y, \hat{\alpha}(y))\big) \leq 0. \tag{3.43}$$

On the other hand, by using λ-convexity of ψ, we also have

$$(\alpha' - \alpha) \cdot (\partial_\alpha \psi(\alpha') - \partial_\alpha \psi(\alpha)) \geq 2\lambda |\alpha' - \alpha|^2.$$

Since $\partial_\alpha H = y + \partial_\alpha \psi$, we also have

$$\begin{aligned} 2\lambda |\hat{\alpha}(y') - \hat{\alpha}(y)|^2 \leq & (\hat{\alpha}(y') - \hat{\alpha}(y)) \cdot (\partial_\alpha \psi(\hat{\alpha}(y')) - \partial_\alpha \psi(\hat{\alpha}(y))) \\ \leq & \big(\hat{\alpha}(y') - \hat{\alpha}(y)\big) \cdot \big(\partial_\alpha H(t, x, \mu, y', \hat{\alpha}(y')) - \partial_\alpha H(t, x, \mu, y, \hat{\alpha}(y))\big) \\ & + \big(\hat{\alpha}(y') - \hat{\alpha}(y)\big) \cdot (y' - y) \leq |\hat{\alpha}(y') - \hat{\alpha}(y)||y' - y|. \end{aligned}$$

In the last inequality we have used (3.43). It then follows

$$|\hat{\alpha}(y') - \hat{\alpha}(y)| \leq \tfrac{1}{2\lambda} |y' - y|. \tag{3.44}$$

This concludes the proof. □

Remark 3.11. *From the Lemma above we deduce that the minimum $\hat{\alpha}$ of the Hamiltonian H does not depend on ε. On the other hand, the optimal control depends on ε, via the adjoint process: it can be written as follows*

$$\hat{\alpha}_t^\varepsilon := \hat{\alpha}(Y_t^\varepsilon). \tag{3.45}$$

Given the optimal control $\hat{\alpha}_t^\varepsilon$, we can write a *forward backward system of stochastic differential equations* (FB-SDE) of McKean-Vlasov type, that is

$$\begin{cases} \mathrm{d}X_t^\varepsilon = (b(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon)) + \hat{\alpha}(Y_t^\varepsilon))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t, \\ -\mathrm{d}Y_t^\varepsilon = (\nabla_x H(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon), Y_t^\varepsilon, \hat{\alpha}(Y_t^\varepsilon)) + \tilde{\mathbb{E}}[\partial_\mu H(t, \tilde{X}_t^\varepsilon, \mathcal{L}(\tilde{X}_t^\varepsilon), \tilde{Y}_t^\varepsilon, \hat{\alpha}(\tilde{Y}_t^\varepsilon))(X_t^\varepsilon)])\,\mathrm{d}t \\ \qquad\qquad + Z_t^\varepsilon\,\mathrm{d}W_t, \\ X_0^\varepsilon = \xi, \quad Y_T^\varepsilon = \nabla_x g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon)) + \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T^\varepsilon, \mathcal{L}(\tilde{X}_T^\varepsilon))(X_T)], \end{cases} \tag{FB-SDE}$$

where we recall again that the notation $(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon)$ denotes an independent copy of $(X_t^\varepsilon, Y_t^\varepsilon)$ defined on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$. If the coefficients are smooth and no further assumptions are required, systems of FB-SDE are not always solvable, see [3]. For a fixed $\varepsilon > 0$, existence of a solution of (FB-SDE) is provided in [19]. In the next subsection, we show that convexity of the Hamiltonian ensures well-posedness of (FB-SDE), even when the viscosity coefficient is zero. Thus, stability estimates on the solutions of (FB-SDE) will turn into an ε-uniform bound on the Lipschitz constant of the optimal control of $(\mathcal{P}_\varepsilon)$, as we will show in Section 4.

3.2. **Well-posedness of (FB-SDE).** The goal of this subsection is to prove well-posedness of the system (FB-SDE) associated to an optimal control, and then to build the associated decoupling field. Note that Theorem 3.6 ensures that solving (SOC) is equivalent to solve the system (FB-SDE). We drop the ε superscript in the whole subsection, for simplicity of notations.

We adopt the strategy known as *continuation method* for FB-SDEs, see [42]. We denote by $\Theta_t := (X_t, \mathcal{L}(X_t), Y_t, \hat{\alpha}_t)$, where $\hat{\alpha}_t = \hat{\alpha}(Y_t)$, and $\mathscr{S}$ is the space of the processes Θ_t such that

$$\|\Theta\|_{\mathscr{S}} := \mathbb{E}\left[\sup_{t\in(0,T)} \left(|X_t|^2 + |Y_t|^2\right) + \int_0^T \left(|Z_t|^2 + |\hat{\alpha}_t|^2\right) \mathrm{d}t\right]^{1/2} < +\infty, \tag{3.46}$$

where Z_t is the process associated to Y_t as in (FB-SDE). Similarly, we define $\theta_t := (X_t, \mathcal{L}(X_t))$. Moreover, an *input* $\mathcal{I} = (I_t^b, I_t^\sigma, I_t^f, I_T^g)$ will be a four-tuple where the first three entries are square-integrable progressively measurable processes and the last one is an $\mathcal{F}_T$ square-integrable random variable. We denote by $\mathbb{I}$ the space of inputs endowed with the norm

$$\|I\|_{\mathbb{I}} := \mathbb{E}\left[|I_T^g|^2 + \int_0^T \left(|I_t^b|^2 + |I_t^\sigma|^2 + |I_t^f|^2\right) \mathrm{d}t\right]^{1/2} < +\infty. \tag{3.47}$$

Definition 3.12. For each $\gamma \in [0,1], \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $I \in \mathbb{I}$, define $\mathcal{E}(\gamma, \xi, I)$ as the FB-SDE:

$$\begin{cases} \mathrm{d}X_t = \left(\gamma[b(t,\theta_t) + \hat{\alpha}_t] + I_t^b\right) \mathrm{d}t + \left(\gamma\sqrt{2\varepsilon} + I_t^\sigma\right) \mathrm{d}W_t, \\ \mathrm{d}Y_t = -\left(\gamma\left\{\nabla_x H(t,\Theta_t) + \tilde{\mathbb{E}}\left[\partial_\mu H(t,\tilde{\Theta}_t)(X_t)\right]\right\} + I_t^f\right) \mathrm{d}t + Z_t\,\mathrm{d}W_t, \\ X_0 = \xi, \\ Y_T = \gamma\left\{\nabla g(X_T, \mathcal{L}(X_T)) + \tilde{\mathbb{E}}\left[\partial_\mu g(\tilde{X}_T, \mathcal{L}(\tilde{X}_T))(X_T)\right]\right\} + I_T^g. \end{cases} \tag{3.48}$$

For any $\gamma \in [0,1]$, we say that the property (S_γ) holds if, for any $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $I \in \mathbb{I}$, the FB-SDE $\mathcal{E}(\gamma, \xi, I)$ has a unique solution in $\mathscr{S}$.

We now provide a stability lemma for solutions of (3.48).

Lemma 3.13. *Let $\gamma \in [0,1]$ such that (S_γ) holds. Then, there exists a constant C, which depends on T, L, λ and it is independent on γ and ε, such that for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $I, I' \in \mathbb{I}$, the solutions Θ, Θ' of $\mathcal{E}(\gamma, \xi, I), \mathcal{E}(\gamma, \xi', I')$ satisfy:*

$$\|\Theta - \Theta'\|_{\mathscr{S}}^2 \leq C\left(\mathbb{E}\left[|\xi - \xi'|^2\right] + \|I - I'\|_{\mathbb{I}}^2\right). \tag{3.49}$$

Proof. The proof strongly relies on the estimates (and the strategies) proved in Lemma 3.2 and Lemma 3.5. For the forward component X_t we have

$$\mathbb{E}\left[\sup_{t\in(0,T)} |X_t - X_t'|^2\right] \leq \mathbb{E}\left[|\xi - \xi'|^2\right] + C\gamma\mathbb{E}\left[\int_0^T |\hat{\alpha}(Y_t) - \hat{\alpha}(Y_t')|^2 \,\mathrm{d}t\right] + C\|I - I'\|_{\mathbb{I}}^2, \tag{3.50}$$

while for the backward component Y_t, Z_t it holds

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in(0,T)} |Y_t - Y_t'|^2 + \int_0^T |Z_t - Z_t'|^2 \,\mathrm{d}t\right] &\leq C\gamma\mathbb{E}\left[\sup_{t\in(0,T)} |X_t - X_t'|^2 + \int_0^T |\hat{\alpha}(Y_t) - \hat{\alpha}(Y_t')|^2 \,\mathrm{d}t\right] \\ &\quad + C\|I - I'\|_{\mathbb{I}}^2. \end{aligned} \tag{3.51}$$

By plugging (3.50) in (3.51) and using Lipschitz continuity of the optimal control proved in Lemma 3.10, we obtain that

$$\begin{aligned} \mathbb{E}\left[|Y_t - Y_t'|^2\right] &\leq C\left(\mathbb{E}\left[|\xi - \xi'|^2\right] + \mathbb{E}\left[\int_0^T |\hat{\alpha}(Y_t) - \hat{\alpha}(Y_t')|^2 \,\mathrm{d}t\right] + \|I - I'\|_{\mathbb{I}}^2\right) \\ &\leq C\left(\mathbb{E}\left[|\xi - \xi'|^2\right] + \|I - I'\|_{\mathbb{I}}^2\right) + \frac{C}{\lambda^2}\int_0^T \mathbb{E}\left[|Y_t - Y_t'|^2\right] \mathrm{d}t, \end{aligned}$$

which yields to

$$\mathbb{E}\left[\sup_{t\in(0,T)}|Y_t-Y_t'|^2\right]\le e^{\frac{C}{\lambda^2}T}C\left(\mathbb{E}\left[|\xi-\xi'|^2\right]+\|I-I'\|_{\mathbb{I}}^2\right). \tag{3.52}$$

We now focus on (3.50): we use Lipschitz continuity of the optimal control, together with (3.52), to obtain

$$\mathbb{E}\left[\sup_{t\in(0,T)}|X_t-X_t'|^2\right]\le \frac{C}{\lambda^2}Te^{\frac{C}{\lambda^2}T}C\left(\mathbb{E}\left[|\xi-\xi'|^2\right]+\|I-I'\|_{\mathbb{I}}^2\right). \tag{3.53}$$

Finally, from (3.51), (3.52), and (3.53) we conclude that

$$\mathbb{E}\left[\int_0^T|Z_t-Z_t'|^2\,\mathrm{d}t\right]\le C\left(\mathbb{E}\left[|\xi-\xi'|^2\right]+\|I-I'\|_{\mathbb{I}}^2\right). \tag{3.54}$$

Then, the result immediately follows, by using again Lipschitz continuity of $\hat{\alpha}$ and the above estimates. □

We now give an *induction* lemma for the system (3.48).

Lemma 3.14. *There exists a $\delta_0>0$, which depends on T,L,λ only, such that, if (S_γ) holds for some $\gamma\in[0,1)$, then $(S_{\gamma+\eta})$ holds for all $\eta\in(0,\delta_0]$ satisfying $\gamma+\eta\le 1$.*

Proof. The proof follows a standard Picard's contraction argument. Indeed, if γ is such that $(\mathcal{S}_\gamma)$ holds, for $\eta>0,\xi\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};\mathbb{R}^d)$ and $I\in\mathbb{I}$, we define the map $\Phi:\mathscr{S}\to\mathscr{S}$ whose fixed points coincide with the solution of $\mathcal{E}(\gamma+\eta,\xi,I)$. We now give the definition of Φ. Given a process $\Theta\in\mathscr{S}$, we denote with Θ' the solution of $\mathcal{E}(\gamma,\xi,I')$ with

$$\begin{aligned}
I_t^{b,'}&=\eta b(t,\theta_t)+\eta\hat{\alpha}(Y_t)+I_t^b\\
I_t^{f,'}&=\eta\nabla H(t,\Theta_t)+\eta\tilde{\mathbb{E}}\left[\partial_\mu H(t,\tilde{\Theta}_t)(X_t)\right]+I_t^f\\
I_t^{\sigma,'}&=\eta\sqrt{2\varepsilon}+I_t^\sigma\\
I_T^{g,'}&=\eta\nabla g(X_T,\mathcal{L}(X_T))+\eta\tilde{\mathbb{E}}\left[\partial_\mu g(\tilde{X}_T,\mathcal{L}(\tilde{X}_T))(X_T)\right]+I_T^g.
\end{aligned}$$

By assumptions, it is uniquely defined and it belongs to $\mathscr{S}$, so the mapping $\Phi:\Theta\to\Theta'$ maps $\mathscr{S}$ into itself. It is clear that a process Θ is a fixed point for Φ if and only if Θ is a solution of $\mathcal{E}(\gamma,\xi,I')$. We now only need to prove that Φ is a contraction when η is small enough. Given $\Theta^1,\Theta^2\in\mathscr{S}$, by Lemma 3.13 we get that

$$\|\Phi(\Theta^1)-\Phi(\Theta^2)\|_{\mathscr{S}}\le C\eta\|\Theta^1-\Theta^2\|_{\mathscr{S}}, \tag{3.55}$$

which is enough to conclude the proof, since the constant C does not depend on γ. □

We are now able to prove well-posedness of (FB-SDE).

Theorem 3.15. *Assume that* **Assumptions (A), (B), (C), (J)** *hold. Then, for any initial $\xi\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$, the system* (FB-SDE) *is uniquely solvable.*

Proof. First, note that for $\gamma=0$, the right hand side of the system (3.48) is made up of square-integrable progressively measurable processes, and it does not depends on the solution itself. So (S_0) obviously holds. Then, the proof is a straightforward induction argument based on Lemma 3.14. □

As already stressed above, by Theorem 3.15 we know that the solution of (FB-SDE) is the unique optimal path of the stochastic control problem (SOC). To conclude this section, we show the existence of a decoupling field related to the system (FB-SDE), which will allow to write the optimal control $\hat{\alpha}_t^\varepsilon$ in feedback form.

Lemma 3.16. *For any $t\in[0,T]$ and $\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, there exists a unique solution*

$$(X_{t,s}^{\xi,\varepsilon},Y_{t,s}^{\xi,\varepsilon},Z_{t,s}^{\xi,\varepsilon})_{t\le s\le T}$$

of the system (FB-SDE) *on* $[t,T]$ *with* $X_{t,t}^{\xi,\varepsilon}=\xi$. *Moreover, for any* $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, *there exists a measurable mapping* $\mathcal{U}^\varepsilon(t,\cdot,\mu):x\in\mathbb{R}^d\mapsto\mathcal{U}^\varepsilon(t,x,\mu)$ *such that:*

$$\mathbb{P}\left[Y_{t,t}^{\xi,\varepsilon}=\mathcal{U}^\varepsilon(t,\xi,\mathcal{L}(\xi))\right]=1. \tag{3.56}$$

Moreover, there exists a constant C, *depending only on the parameters in* **Assumptions (A), (B), (C), (J)**, *such that, for any* $t\in[0,T]$ *and any* $\xi_1,\xi_2\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$,

$$\mathbb{E}\left[|\mathcal{U}(t,\xi_1,\mathcal{L}(\xi_1))-\mathcal{U}(t,\xi_2,\mathcal{L}(\xi_2))|^2\right]\le C\mathbb{E}\left[|\xi_1-\xi_2|^2\right]. \tag{3.57}$$

Proof. Given $t\in[0,T)$ and $\xi\in L^2(\Omega,\mathscr{F}_t,\mathbb{P};\mathbb{R}^d)$, existence and uniqueness of a solution of (FB-SDE) on $[t,T]$ with initial condition ξ is a direct consequence of Theorem 3.15. We now proceed to define the decoupling field. First of all, note that $Y_{t,t}^{\xi,\varepsilon}$ coincide a.s. with a $\sigma\{\xi\}$-measurable $\mathbb{R}^d$-valued random variable. In particular, there exists $u_\xi^\varepsilon(t,\cdot):\mathbb{R}^d\to\mathbb{R}^d$ such that $\mathbb{P}\left[Y_{t,t}^{\xi,\varepsilon}=u_\xi^\varepsilon(t,\xi)\right]=1$. Moreover, the law of $(\xi,Y_{t,t}^{\xi,\varepsilon})$ only depends on the law of ξ, as a consequence of Yamada-Watanabe Theorem, see [22, Theorem 1.33]. Since uniqueness holds pathwise, it also holds in law, so given two initial conditions with the same law, the solution has the same law. Therefore, given another $\mathbb{R}^d$-valued random vector ξ' with the same law of ξ, it holds that $(\xi,u_\xi^\varepsilon(t,\xi))$ has the same law of $(\xi',u_{\xi'}^\varepsilon(t,\xi'))$. In particular, for any measurable vector field $v:\mathbb{R}^d\to\mathbb{R}^d$, the random variables $u_\xi^\varepsilon(t,\xi)-v(\xi)$ and $u_{\xi'}^\varepsilon(t,\xi')-v(\xi')$ have the same law. Choosing $v=u_\xi^\varepsilon(t,\cdot)$ we deduce that $u_\xi^\varepsilon(t,\cdot)$ and $u_{\xi'}^\varepsilon(t,\cdot)$ are equal a.e. under the same law $\mathcal{L}(\xi)$. This means that, by denoting $\mathcal{L}(\xi)=\mu$, there exists an element $\mathcal{U}^\varepsilon(t,\cdot,\mu)\in L^2(\mathbb{R}^d;\mu)$ such that both $u_\xi^\varepsilon(t,\cdot)$ and $u_{\xi'}^\varepsilon(t,\cdot)$ coincide with $\mathcal{U}^\varepsilon(t,\cdot,\mu)$. Identifying $\mathcal{U}^\varepsilon(t,\cdot,\mu)$ with one of its versions, we have that $\mathbb{P}\left[Y_{t,t}^{\xi,\varepsilon}=\mathcal{U}^\varepsilon(t,\xi,\mu)\right]=1$. When $t>0$, for any $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ there exists a $\mathcal{F}_t$-measurable random variable ξ with law μ. As a consequence, this procedure allows to define $\mathcal{U}^\varepsilon(t,\cdot,\mu)$ for any $\mu\in\mathscr{P}_2(\mathbb{R}^d)$. When $t=0$, $\mathcal{F}_0$ may reduce to events of measure zero or one. In such a case, $\mathcal{F}_0$ can be enlarged, with no loss of generality, to support $\mathbb{R}^d$-valued random variables with arbitrary distributions.

The fact that $\mathcal{U}^\varepsilon$ is independent from the probabilistic set-up $(\Omega,\mathcal{F}_t,\mathbb{P})$ directly follows from the uniqueness in law.

Finally, the Lipschitz property of $\mathcal{U}^\varepsilon(0,\cdot,\cdot)$ is a consequence of Lemma 3.13 with $\gamma=1$. Shifting time if necessary, the same argument applies to $\mathcal{U}^\varepsilon(t,\cdot,\cdot)$. □

Remark 3.17. *It is worth to notice that the decoupling fields are different if the laws of the initial conditions are different.*

4. The vanishing viscosity method

In this section we prove our main result. We first build the optimal control for $(\mathcal{P}_\varepsilon)$ using the theory developed in Section 3. Then, we will provide some convergence lemma, which will be the core of the proof of Theorem 1.1. To make the presentation smoother, we will always assume that **Assumptions (A), (B), (C), (J)** hold, without recalling them.

4.1. **The viscous optimal control.** Let $\mathcal{U}^\varepsilon$ be the decoupling field given by Lemma 3.16. Thanks to Proposition 2.10, for any $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, we can consider a version of $x\mapsto\mathcal{U}^\varepsilon(t,x,\mu)$ in $L^2(\mathbb{R}^d,\mu)$ that is Lipschitz continuous with respect to x, for the same Lipschitz constant C as in (3.57). This is crucial for what follows.

Lemma 4.1. *Let* $\hat\alpha$ *be the minimizer of the Hamiltonian* (3.11) *in* U, *and* $\mathcal{U}^\varepsilon$ *the decoupling field defined in Lemma 3.16. Then, the map* $u^\varepsilon:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ *defined by*

$$u^\varepsilon(t,x)=\hat\alpha(\mathcal{U}^\varepsilon(t,x,\mu_t^\varepsilon)), \tag{4.1}$$

is the unique optimal control for $(\mathcal{P}_\varepsilon)$. *Moreover, the control* u^ε *is Lipschitz continuous, uniformly with respect to time and viscosity coefficient: i.e., there exists a constant* $L_\lambda>0$ *independent on* t *and* ε *such that*

$$|u^\varepsilon(t,x)-u^\varepsilon(t,x')|\le L_\lambda|x-x'|. \tag{4.2}$$

Proof. First remark that Lipschitz continuity follows from Lemma 3.10 and Lemma 3.16: it holds

$$|u^\varepsilon(t,x) - u^\varepsilon(t,x')| = |\hat{\alpha}(\mathcal{U}^\varepsilon(t,x,\mu_t^\varepsilon)) - \hat{\alpha}(\mathcal{U}^\varepsilon(t,x',\mu_t^\varepsilon))| \leq L_\lambda |x - x'|, \tag{4.3}$$

for some constant L_λ which only depends on the norms of b, f, g and on λ. We now prove optimality. Let $w : [0,T] \times \mathbb{R}^d \to U$ be an admissible control for $(\mathcal{P}_\varepsilon)$. Then, the vector field α_t defined as

$$\alpha_t = w(t, X_t^{w,\varepsilon})$$

is an admissible control for (SOC) where $X_t^{w,\varepsilon}$ is the associated trajectory. With this particular choice, the associated cost functional can be rewritten as

$$J^S(\boldsymbol{\alpha}) = \int_0^T \int_{\mathbb{R}^d} (f(t,x,\mu_t^{w,\varepsilon}) + \psi(w(t,x)))\, \mu_t^{w,\varepsilon}(\,\mathrm{d}x)\,\mathrm{d}t + \int_{\mathbb{R}^d} g(x, \mu_T^{w,\varepsilon}) \mu_T^{w,\varepsilon}(\,\mathrm{d}x) = J(\mu^{w,\varepsilon}, w),$$

where $\mu_t^{w,\varepsilon}$ is the law $\mathcal{L}(X_t^{w,\varepsilon})$ of the controlled trajectory $X_t^{w,\varepsilon}$. Then, by the strict minimality property of $\hat{\alpha}_t^\varepsilon$ for (SOC), it holds

$$\begin{aligned}
J(\mu^\varepsilon, u^\varepsilon) = &\int_0^T \int_{\mathbb{R}^d} (f(t,x,\mu_t^\varepsilon) + \psi(u^\varepsilon(t,x)))\, \mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t + \int_{\mathbb{R}^d} g(x,\mu_T^\varepsilon)\mu_T^\varepsilon(\,\mathrm{d}x) \\
=&\mathbb{E}\left[g(X_T^\varepsilon, \mathcal{L}(X_T^\varepsilon)) + \int_0^T (f(t, X_t^\varepsilon, \mathcal{L}(X_t^\varepsilon)) + \psi(\hat{\alpha}_t^\varepsilon))\,\mathrm{d}t\right] \\
<&\mathbb{E}\left[g(X_T^{w,\varepsilon}, \mathcal{L}(X_T^{w,\varepsilon})) + \int_0^T (f(t, X_t^{w,\varepsilon}, \mathcal{L}(X_t^{w,\varepsilon})) + \psi(\alpha_t))\,\mathrm{d}t\right] \\
=&\int_0^T \int_{\mathbb{R}^d} (f(t,x,\mu_t^{w,\varepsilon}) + \psi(w(t,x)))\, \mu_t^{w,\varepsilon}(\,\mathrm{d}x)\,\mathrm{d}t + \int_{\mathbb{R}^d} g(x,\mu_T^{w,\varepsilon})\mu_T^{w,\varepsilon}(\,\mathrm{d}x),
\end{aligned}$$

for any admissible control w. This proves the optimality of u^ε and concludes the proof. □

4.2. **Convergence lemmas.** In this section we prove a series of useful convergence estimates, that will be the key tools to prove our main theorem.

Lemma 4.2. *Let $K \subset \mathbb{R}^d$ be a bounded set and $C_K > 0$ a fixed constant. Define*

$$\mathcal{A}_K := \{u \in L^2((0,T); W^{1,\infty}(K,U)) : \sup_{t\in(0,T)} \|u(t,\cdot)\|_{W^{1,\infty}(K,U)} \leq C_K\}. \tag{4.4}$$

Then, $\mathcal{A}_K$ is compact in the weak $L^2((0,T); W^{1,p}(K,U))$-topology for any $p \in (1,\infty)$.

Proof. See e.g. [33, Theorem 2.5] □

We have the following convergence result for the sequence u^ε defined in (4.1).

Corollary 4.3 (Convergence of the controls)**.** *Let u^ε be the sequence of optimal controls given by* (4.1)*. Then, there exists a map $u \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}^d,U))$ such that, for every $1 < p < \infty$, the following convergence holds*

$$u^\varepsilon \rightharpoonup u \quad \textit{in } L^2((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d,U)). \tag{4.5}$$

Proof. The result is a direct application of Lemma 4.2 together with Lemma 4.1. The constant C_K appearing in Lemma 4.2 is chosen as $\max\{L_\lambda, R\}$ where L_λ is the constant in (4.2) and R is defined in (A1). In particular, the constant C_K does not depend on ε. □

We now show the convergence of the optimal trajectories.

Lemma 4.4 (Convergence of the trajectories)**.** *Let u, u^ε be given by Lemma 4.3 and let μ, μ^ε be the unique solution of the deterministic* (1.2) *and the viscous equation* (1.3) *with vector field b, and control u, u^ε respectively. It then holds*

$$\lim_{\varepsilon\to 0} \sup_{t\in(0,T)} W_2(\mu_t^\varepsilon, \mu_t) = 0. \tag{4.6}$$

Proof. We divide the proof in two steps.

<u>*Step 1*</u> **Compactness of the sequence μ^ε.** We start by proving compactness of $\{\mu^\varepsilon\}_{\varepsilon>0}$ in $C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ as a consequence of Ascoli-Arzelà's Theorem. First of all, we exploit a uniform bound on the second moment of μ^ε_t: since $\mu^\varepsilon_t \in \mathscr{P}_2(\mathbb{R}^d)$, one can use $|x|^2$ as a test function in the equation, obtaining

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_t(\,\mathrm{d}x) &= \int_{\mathbb{R}^d}|x|^2\mu_0(\,\mathrm{d}x) + 2\int_0^t\int_{\mathbb{R}^d}u^\varepsilon(s,x)\cdot x\,\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s\\ &\quad + 2\int_0^t\int_{\mathbb{R}^d}b(s,x,\mu^\varepsilon_s)\cdot x\,\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s + 2\varepsilon t.\end{aligned}$$

For the term involving the control, from (A1) we easily get

$$\begin{aligned}2\left|\int_0^t\int_{\mathbb{R}^d}u^\varepsilon(s,x)\cdot x\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s\right| &\le 2R\int_0^t\int_{\mathbb{R}^d}|x|\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s\\ &\le 2RT + 2R\int_0^t\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s.\end{aligned}$$

On the other hand, from (B2) and Young inequality we get

$$\begin{aligned}2\left|\int_0^t\int_{\mathbb{R}^d}b(s,x,\mu^\varepsilon_s)\cdot x\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s\right| &\le 2M\int_0^t\int_{\mathbb{R}^d}|x|\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s + 2M\int_0^t\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s\\ &\quad + \int_0^t\left(\int_{\mathbb{R}^d}|x|\mu^\varepsilon_s(\,\mathrm{d}x)\right)^2\,\mathrm{d}s\\ &\le 2MT + 5M\int_0^t\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s.\end{aligned}$$

Thus, being $0<\varepsilon<1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_t(\,\mathrm{d}x) &\le \int_{\mathbb{R}^d}|x|^2\mu_0(\,\mathrm{d}x) + 2(1+M+R)T\\ &\quad + (2R+5M)\int_0^t\int_{\mathbb{R}^d}|x|^2\mu^\varepsilon_s(\,\mathrm{d}x)\,\mathrm{d}s,\end{aligned}$$

and then Gronwall's lemma gives that

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^d}|x|^2\,\mu^\varepsilon_t(\,\mathrm{d}x) \le [M_2(\mu_0) + 2(1+M+R)T]\,e^{(2R+5M)T}, \tag{4.7}$$

providing a uniform bound on $M_2(\mu^\varepsilon_t)$. This means that the sequence $\{\mu^\varepsilon\}_{\varepsilon>0}$ takes values in a relatively compact set in $\mathscr{P}_2(\mathbb{R}^d)$ (endowed with W_2). Next, we show that the family $\{\mu^\varepsilon\}_{\varepsilon>0}$ is equi-continuous in $C([0,T];\mathscr{P}_2(\mathbb{R}^d))$. Let X^ε_t be a solution of (3.1) with law μ^ε_t, then by Proposition 2.6 we have that

$$W_2(\mu^\varepsilon_t,\mu^\varepsilon_s)^2 \le \mathbb{E}\left[|X^\varepsilon_t - X^\varepsilon_s|^2\right].$$

By using equation (3.1) we compute

$$\begin{aligned}W_2(\mu^\varepsilon_t,\mu^\varepsilon_s)^2 &\le \mathbb{E}\left[|X^\varepsilon_t - X^\varepsilon_s|^2\right]\\ &\le 4T\int_s^t\mathbb{E}\left[|b(\tau,X^\varepsilon_\tau,\mu^\varepsilon_\tau)|^2 + |\alpha^\varepsilon_\tau|^2\right]\,\mathrm{d}\tau + 4\varepsilon\mathbb{E}\left[|W_t - W_s|^2\right]\\ &\le 4T\left(M^2\int_s^t[1+|X^\varepsilon_\tau|^2 + M_2(\mu^\varepsilon_\tau)^2]\,\mathrm{d}\tau + R^2|t-s| + c(d)|t-s|\right)\\ &\le C(\mu_0,T,M,R)|t-s|,\end{aligned}$$

which implies equi-continuity of the sequence $\{\mu^\varepsilon\}_{\varepsilon>0}$ in $C([0,T];\mathscr{P}_2(\mathbb{R}^d))$. Since $\mathscr{P}_2(\mathbb{R}^d)$ is a complete metric space [50], by Ascoli-Arzelà's Theorem (see [2, Proposition 3.3.1]) the sequence

$\{\mu^\varepsilon\}_{\varepsilon>0}$ is relatively compact in $C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ for every $T>0$. Then, up to a sub-sequence that we do not relabel, there exists a probability measure $\rho \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$ such that

$$\mu^\varepsilon \to \rho \ \text{ in } C([0,T];\mathscr{P}_2(\mathbb{R}^d)), \tag{4.8}$$

which means that

$$\lim_{\varepsilon\to 0} \sup_{t\in(0,T)} W_2(\mu_t^\varepsilon,\rho_t) = 0. \tag{4.9}$$

Step 2 **Identification of the limit.** In this step we show that ρ is a solution of (1.2). This will imply that, by uniqueness, it holds $\rho=\mu$ and the whole sequence μ^ε converges to μ. Let $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$: by Definition 2.15 we have that

$$\int_0^T\int_{\mathbb{R}^d}\Big(\partial_t\varphi(t,x)+(b(t,x,\mu_t^\varepsilon)+u^\varepsilon(t,x))\cdot\nabla\varphi(t,x)+\varepsilon\Delta\varphi(t,x)\Big)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t=\int_{\mathbb{R}^d}\varphi(0,x)\mu_0(\,\mathrm{d}x).$$

Notice that by Proposition 2.3 we know that (4.8) implies weak convergence, thus it holds

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}\Big(\partial_t\varphi(t,x)+\varepsilon\Delta\varphi(t,x)\Big)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t=\int_0^T\int_{\mathbb{R}^d}\partial_t\varphi(t,x)\rho_t(\,\mathrm{d}x)\,\mathrm{d}t.$$

We now consider the term involving the control. Denote by $K:=\operatorname{supp}(\varphi)$ and $C_K:=\|u^\varepsilon\|_{L^\infty((0,T);L^\infty(K))}$. It then holds

$$\operatorname{Lip}(\varphi(t,\cdot)u^\varepsilon(t,\cdot))\le L\|\varphi\|_{L^\infty}+C_K\|\nabla\varphi\|_{L^\infty}:=C_\varphi.$$

Then, by Lemma 2.4, it holds

$$\limsup_{\varepsilon}\left|\int_0^T\int_{\mathbb{R}^d}\varphi(t,x)u^\varepsilon(t,x)\Big(\mu_t^\varepsilon(\,\mathrm{d}x)-\rho_t(\,\mathrm{d}x)\Big)\,\mathrm{d}t\right|\le C_\varphi\limsup_{\varepsilon}\int_0^T W_1(\mu_t^\varepsilon,\rho_t)\,\mathrm{d}t, \tag{4.10}$$

which converges to 0 as $\varepsilon\to 0$, by using (4.9) and (2.4). On the other hand, it holds

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}\varphi(t,x)u^\varepsilon(t,x)\rho_t(\,\mathrm{d}x)\,\mathrm{d}t=\int_0^T\varphi(t,x)u(t,x)\rho_t(\,\mathrm{d}x)\,\mathrm{d}t,$$

due to convergence in (4.5) and the fact that $\varphi\rho$ belongs to $L^\infty((0,T);W^{-1,p'}(\mathbb{R}^d,\mathbb{R}^d))$ and has compact support. Then, we have shown

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}u^\varepsilon(t,x)\cdot\nabla\varphi(t,x)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t=\int_{\mathbb{R}^d}u(t,x)\cdot\nabla\varphi(t,x)\rho_t(\,\mathrm{d}x)\,\mathrm{d}t.$$

We are left to prove convergence in the non-linear term: on one hand the convergence (4.8) implies that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d}b(t,x,\rho_t)\cdot\nabla\varphi(t,x)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t=\int_0^T\int_{\mathbb{R}^d}b(t,x,\rho_t)\cdot\nabla\varphi(t,x)\rho_t(\,\mathrm{d}x)\,\mathrm{d}t. \tag{4.11}$$

On the other hand, the uniform Lipschitz assumption (B3) on b implies that

$$\left|\int_0^T\int_{\mathbb{R}^d}(b(t,x,\mu_t^\varepsilon)-b(t,x,\rho_t))\cdot\nabla\varphi(t,x)\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t\right|\le L\|\nabla\varphi\|_{L^\infty}\sup_{t\in(0,T)}W_2(\mu_t^\varepsilon,\rho_t), \tag{4.12}$$

which converges to 0 when $\varepsilon\to 0$ thanks to the convergence in (4.9). Hence, ρ_t is a solution of (1.2) and this concludes the proof. □

Lemma 4.5 (Convergence of the cost)**.** *Let u,u^ε be given by Lemma 4.3 and let μ,μ^ε be the corresponding unique solution of the deterministic* (1.2) *and the viscous equation* (1.3) *with vector field b, and control u,u^ε respectively. Let w be an admissible control for $(\mathcal{P}_\varepsilon)$: then we have*

$$\lim_{\varepsilon\to 0}J(\mu^\varepsilon,w)=J(\mu,w). \tag{4.13}$$

Moreover, if u^ε is the sequence of optimal controls as in (4.3)*, we have that*

$$J(\mu,u)\le\liminf_{\varepsilon\to 0}J(\mu^\varepsilon,u^\varepsilon). \tag{4.14}$$

Proof. We divide the proof in two steps.

Step 1 **Convergence for a fixed control.** First, convergence of the control cost immediately follows from Lemma 4.4. We now analyze the running cost, the same argument also applies to the final cost. By (J2), we have

$$\left|\int_0^T \int_{\mathbb{R}^d} \left(f(t,x,\mu_t^\varepsilon) - f(t,x,\mu_t)\right) \mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t\right| \le L \sup_{t\in(0,T)} M_2(\mu_t^\varepsilon) \sup_{t\in(0,T)} W_2(\mu_t^\varepsilon,\mu_t). \tag{4.15}$$

The conclusion follows from (4.9), since $M_2(\mu_t^\varepsilon)$ is uniformly bounded in t and ε.

Step 2 **Semi-continuity.** It follows from [33, Theorem 2.12]: arguing as in Step 1, we can show convergence of both running and final costs. Then, we must show that the control cost is lower semi-continuous with respect to the weak convergence (4.5). First of all, by Theorem 2.14 we can fix $r > 0$ such that $\operatorname{supp}\mu_t \subset B_r$ for all $t \in [0,T]$. Let $p > d$ and define the functional $S^\mu : L^2((0,T); W^{1,p}(B_r)) \to [0,+\infty]$ as

$$S^\mu(g) := \begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}^d} \psi(g(t,x))\mu_t(\,\mathrm{d}x)\,\mathrm{d}t, & \text{if } \mathrm{Lip}(g(t,\cdot)) \in L^\infty(0,T),\\ +\infty & \text{otherwise.}\end{cases} \tag{4.16}$$

By convexity of ψ, it is immediate to check that S^μ is convex: thus, it is sufficient to show that it is lower semi-continuous in the strong topology $L^2((0,T); W^{1,p}(B_r))$ to obtain weak lower semi-continuity. Let g_k be a sequence in $L^2((0,T); W^{1,p}(B_r))$ strongly converging to some g. By using (J1), we have

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^d} \left(\psi(g_k(t,x)) - \psi(g(t,x))\right) \mu_t(\,\mathrm{d}x)\,\mathrm{d}t\right| \le \int_0^T \int_{\mathbb{R}^d} |\psi(g_k(t,x)) - \psi(g(t,x))|\, \mu_t(\,\mathrm{d}x)\,\mathrm{d}t\\
&\le L \int_0^T \int_{\mathbb{R}^d} \left(|g_k(t,x)| + |g(t,x)|\right) |g_k(t,x) - g(t,x)|\, \mu_t(\,\mathrm{d}x)\,\mathrm{d}t\\
&\le CL \int_0^T \int_{\mathbb{R}^d} |g_k(t,x) - g(t,x)|^2\, \mu_t(\,\mathrm{d}x)\,\mathrm{d}t \le CL \int_0^T \|g_k(t,\cdot) - g(t,\cdot)\|_{L^\infty}^2\\
&\le CL \int_0^T \|g_k(t,\cdot) - g(t,\cdot)\|_{W^{1,p}}^2,
\end{aligned}$$

where the constant C depends on Sobolev embeddings and the $L^2W^{1,p}$ norm of g_k, g. Therefore, it holds

$$|S^\mu(g_k) - S^\mu(g)| \le CL\|g_k - g\|_{L^2W^{1,p}}^2, \tag{4.17}$$

which gives continuity with respect to the strong topology. Thus, S^μ is weakly lower semi-continuous and by using Corollary 4.3 we obtain that

$$\int_0^T \int_{\mathbb{R}^d} \psi(u(t,x))\mu_t(\,\mathrm{d}x)\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^d} \psi(u^\varepsilon(t,x))\mu_t(\,\mathrm{d}x)\,\mathrm{d}t. \tag{4.18}$$

Finally, observe that ψ is Lipschitz, since it is C^1 on the compact set U. We denote by $L_{\psi,U}$ its Lipschitz constant on U. Moreover, u^ε is Lipschitz, with a Lipschitz constant L_λ independent on ε, as shown in (4.2). Then, it holds:

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^d} \psi(u^\varepsilon(t,x))\left(\mu_t^\varepsilon(\,\mathrm{d}x) - \mu_t(\,\mathrm{d}x)\right)\,\mathrm{d}t\right| \le L_{\psi,U} L_\lambda \int_0^T W_1(\mu_t^\varepsilon,\mu_t)\,\mathrm{d}t\\
&\le L_{\psi,U} L_\lambda \int_0^T W_2(\mu_t^\varepsilon,\mu_t)\,\mathrm{d}t \le C \sup_{t\in(0,T)} W_2(\mu_t^\varepsilon,\mu_t).
\end{aligned}$$

Merging it with (4.18) and recalling (4.9), we have

$$\int_0^T \int_{\mathbb{R}^d} \psi(u(t,x))\mu_t(\,\mathrm{d}x)\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^d} \psi(u^\varepsilon(t,x))\mu_t^\varepsilon(\,\mathrm{d}x)\,\mathrm{d}t. \tag{4.19}$$

□

4.3. **Proof of Theorem 1.1.** We are now ready to prove our main result.

Proof. Let $\Lambda(T, L)$ be the constant in the assumptions of Theorem 1.1. Then, by Lemma 4.1 we know that there exists a unique $(\mu^\varepsilon, u^\varepsilon)$ optimal pair for $(\mathcal{P}_\varepsilon)$. By Lemma 4.3, we know that there exists a function $u \in L^1((0,T); \mathrm{Lip}(\mathbb{R}^d, U))$ such that

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d, U)),$$

for every $1 \leq p < \infty$. This proves point (i) of the theorem. Moreover, there exists a unique $\mu \in C([0,T]; \mathscr{P}_2(\mathbb{R}^d))$ which solves (2.7) with control u; thus, by Lemma 4.4 we have

$$\mu^\varepsilon \to \mu \quad \text{in } C([0,T], \mathscr{P}_2(\mathbb{R}^d)).$$

This is point (ii) of the theorem. The convergence of the cost is a consequence of Lemma 4.5. We only need to show optimality of (μ, u) for $(\mathcal{P})$. Let $w \in \mathcal{A} \setminus \{u\}$ be an admissible control for $(\mathcal{P})$ and μ^w the corresponding trajectory. We define $\mu^{w,\varepsilon}$ to be the unique solution of (1.3) with control w; since $(\mu^{w,\varepsilon}, w)$ is an admissible pair for $(\mathcal{P}_\varepsilon)$ and $(\mu^\varepsilon, u^\varepsilon)$ is the unique optimal pair, we have that

$$J(\mu^\varepsilon, u^\varepsilon) < J(\mu^{w,\varepsilon}, w). \tag{4.20}$$

Now, by Lemma 4.4 we know that $\mu^{w,\varepsilon}$ converges to the unique solution μ^w of (2.7) with control w and the associated cost converges:

$$\lim_{\varepsilon \to 0} J(\mu^{w,\varepsilon}, w) = J(\mu^w, w).$$

Then, combining Lemma 4.5, equation (4.20), and the convergence above, it holds

$$J(\mu, u) \leq \liminf_{\varepsilon \to 0} J(\mu^\varepsilon, u^\varepsilon) \leq J(\mu^w, w),$$

for any admissible pair (μ^w, w). Then, (μ, u) is an optimal pair for $(\mathcal{P})$ and the proof is complete. □

It is important to remark that the lower bound on λ plays a role in the sufficient condition for optimality in Theorem 3.6. However, if one had *a priori* a solution to the problem $(\mathcal{P}_\varepsilon)$, then the convergence theorem holds for a larger class of strictly-convex control cost, without requiring any further assumption on the convexity constant λ. This is obtained by using the necessary condition for optimality [20, Theorem 4.5] to prove Lemma 3.10. Then, the proof of existence (and convergence) of a Lipschitz optimal control can be completed in an analogous way. This remark then leads us to the following corollary.

Corollary 4.6. *Assume that there exists a sequence $\varepsilon_n \to 0$ such that for $\varepsilon_n > 0$ there exists an optimal pair $(u^{\varepsilon_n}, \mu^{\varepsilon_n})$ for $(\mathcal{P}_{\varepsilon_n})$. Assume the following:*

- *the set of admissible control values $U \subset \mathbb{R}^d$ is convex and compact;*
- *the vector field b is $C^{1,1}_{\mathrm{loc}}$ regular, i.e.* **Assumption (B)** *in Section 2.3 above holds;*
- *the functions f, ψ, g in J are $C^{1,1}_{\mathrm{loc}}$ regular, i.e.* **Assumption (J)** *in Section 3.1 above holds;*
- *the function ψ is λ-convex, for some $\lambda > 0$, and the functions f, g are convex, i.e.* **Assumption (C)** *in Section 3.1 above holds.*

Then, there exists a solution $(\mu, u) \in C([0,T]; \mathscr{P}_2(\mathbb{R}^d)) \times L^\infty((0,T); \mathrm{Lip}(\mathbb{R}^d, U))$ of $(\mathcal{P})$ and, up to sub-sequences, the following convergences hold:

(i) *$u^{\varepsilon_n} \rightharpoonup u$ in $L^2((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d, U))$ for every $1 \leq p < \infty$;*
(ii) *$\mu^{\varepsilon_n} \to \mu$ in $C([0,T], \mathscr{P}_2(\mathbb{R}^d))$;*
(iii) *$J(\mu, u) \leq \liminf_{n \to \infty} J(\mu^{\varepsilon_n}, u^{\varepsilon_n})$.*

5. The role of convexity hypotheses

The aim of this section is to discuss the role of the convexity hypotheses **(C)** for the validity of Theorem 1.1. In particular, we show that, by relaxing the strict convexity assumption on the control cost ψ, then convergence of optimal controls u^ε for $(\mathcal{P}_\varepsilon)$ to an optimal control u of $(\mathcal{P})$ is not ensured. This is the core of the counterexample that we describe in the following.

We consider the minimization problem of the functional

$$J(\mu, u) = \int_0^T \int_{\mathbb{R}} \psi(u(t,x))\mu_t(\,\mathrm{d}x)\,\mathrm{d}t \tag{5.1}$$

where the control cost ψ is C^∞, positive, convex and satisfies $\psi(s) = 0$ for $s \in [-1,1]$. Note that the function ψ is clearly not strictly convex. As an example, consider the C^∞, not analytic function

$$\phi(x) = \begin{cases} 0 & \text{for } x = 0, \\ \exp(-1/|x|) & \text{for } x \neq 0. \end{cases}$$

Then, one can build a function $\psi(x)$ as above by choosing

$$\psi(x) := \begin{cases} 0 & \text{for } x \in [0,1], \\ \displaystyle\int_0^{x-1} ds \int_0^s \phi(t)\,dt & \text{for } x > 1, \\ \psi(-x) & \text{for } x < 0. \end{cases}$$

We assume that the dynamics is given by the equation

$$\begin{cases} \partial_t \mu_t + \mathrm{div}[u(t,x)\mu_t] = 0, \\ \mu_0 = \delta_0, \end{cases} \tag{5.2}$$

where δ_0 is the Dirac delta centered in 0.

Set $U = [-1,1]$ and $u \in \mathrm{Lip}(\mathbb{R}; U)$ be an admissible control. By the standard Cauchy-Lipschitz Theorem, to any admissible control we can associate a unique flow X_t, i.e. the solution of

$$\begin{cases} \dot{X}_t = u(X_t), \\ X_0 = x. \end{cases} \tag{5.3}$$

It follows that $\mu_t = \delta_{X_t}$ is the unique solution of (5.2) with control u. It is clear that any Lipschitz function $u \in \mathrm{Lip}(\mathbb{R}; [-1,1])$ is also optimal, since the corresponding cost is identically zero.

We now consider the viscous optimal control, i.e. where the dynamics is governed by the equation

$$\begin{cases} \partial_t \mu_t^\varepsilon + \mathrm{div}[u(t,x)\mu_t^\varepsilon] = \varepsilon \Delta \mu_t^\varepsilon, \\ \mu_0^\varepsilon = \delta_0. \end{cases} \tag{5.4}$$

The same observations made for the non-viscous case apply: every Lipschitz function $u \in \mathrm{Lip}(\mathbb{R}; [-1,1])$ is optimal and it is associated to a unique solution μ^ε of (5.4). For each choice of $u \in \mathrm{Lip}(\mathbb{R}; [-1,1])$, one has convergence of the controls (they are ε-independent), convergence of the trajectories (as an easy consequence of Lemma 4.4), and convergence of the cost (as a consequence of Lemma 4.5). However, the viscous problem has other solutions for which the convergence result does not hold. An example is provided by the function $\tilde{u}(x) = \mathrm{sign}(x)$: its cost is identically zero and thus it is an optimal control. Moreover, $\tilde{u}$ is bounded and then it is associated to a unique solution $\tilde{\mu}^\varepsilon$ of (5.4), see [38]. Nevertheless, since $\tilde{u}$ is independent from ε, it does not converge to a Lipschitz optimal control of the inviscid problem. Furthermore, notice that the equation (5.2) with vector field $\tilde{u}$ has multiple solutions, as a consequence of the non-uniqueness for the corresponding ODE (5.3) with initial datum $x = 0$. As an example, both the trajectories $x(t) = t$ and $x(t) = -t$ are solutions of the ODE in the Caratheodory sense.

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